1 Deriving the Black-Scholes Formula

1.1 Call Option

For a European call option, the potential cash-flows at time t with occur if $S_t > K$

1. Receive a stock worth S_t with probability $\Pr(S_t > K)$
2. Pay K with probability $\Pr(S_t > K)$

$$\begin{aligned}
\text{E(call payoff)} &= \Pr(S_t > K)\ [E(S_t \mid S_t > K) - K] \\
\text{PV}_0[\text{E(call payoff)}] &= e^{-\alpha t} Pr(S_t > K)\ [E(S_t \mid S_t > K) - K]
\end{aligned}$$

1.2 Put Option

For a European put option, the potential cash-flows at time t with occur if $S_t > K$

1. Receive K with probability $\Pr(S_t < K)$
2. Pay (buy a stock worth) S_t with probability $\Pr(S_t < K)$

$$\begin{aligned}
\text{E(put payoff)} &= \Pr(S_t < K)\ [K - E(S_t \mid S_t < K)] \\
\text{PV}_0[\text{E(put payoff)}] &= e^{-\alpha t} Pr(S_t < K)\ [K - E(S_t \mid S_t < K)]
\end{aligned}$$

1.3 Lognormal Model

In order to develop the Black-Scholes formula, we need to know the following quantities

1. $\Pr(S_t > K)$
2. $\text{E}(S_t | S_t > K)$

Let A be the normally distributed random variable for the stock return.

$$S_t = S_0 e^{At} \text{ where } A \sim \text{N}(\alpha, \sigma^2).$$
$$S_t/S_0 \sim LN(m = (\alpha - \delta - 1/2\sigma^2)t, v = \sigma\sqrt{t}).$$

These parameters are chosen s.t. $\mathrm{E}(S_t/S_0) = e^{(\alpha-\delta)t}$ where $\alpha - \delta$ is the capital gains rate. We can see that this is true since $\mathrm{E}(S_t/S_0) = e^{m+1/2v^2} = e^{(\alpha-\delta-1/2\sigma^2)t+1/2\sigma^2 t} = e^{(\alpha-\delta)t}$

For $t = 1$ the volatility of the stock return equals to the volatility of $\ln(S_t/S_0)$
Otherwise, the volatility of $\ln(S_t/S_0)$ must be adjusted for time, so $v = \sigma\sqrt{t}$

$$\begin{aligned} \Pr(S_t < K) &= \Pr(S_t/S_0 < K/S_0) \\ &= \Pr(\ln(S_t/S_0) < \ln(K/S_0)) \end{aligned}$$

Since $\ln(S_t/S_0) \sim \mathrm{Normal}(m, v^2)$, then $(\ln(S_t/S_0) - m)/v = Z \sim \mathrm{N}(0,\ 1)$ where Z is the standard normal random variable. Therefore,

$$\begin{aligned} \Pr(S_t < K) &= \Pr(Z < \frac{\ln(K/S_0) - m}{v}) \\ &= \Pr(Z < -d_2) \\ &= N(-d_2) \end{aligned}$$

where $d_2 = \frac{\ln(S_0/K)+m}{v} = \frac{\ln(S_0/k)+(\alpha-\delta-1/2\sigma^2)}{\sigma\sqrt{t}}$

Since $\Pr(S_t < K) = N(-d_2)$ then

$$\Pr(S_t > K) = N(d_2)$$

To find $\mathrm{E}(S_t|\ S_t < K)$ we use the following formula

$$\mathrm{E}(S_t|\ S_t < K) = \mathrm{PE}(S_t|\ S_t < K)/\Pr(S_t < K)$$

where PE is the partial expectation from $S_t = 0$ to $S_t = K$. Note that

$$\mathrm{PE}(S_t/S_0|\ S_t/S_0 < K/S_0) = \mathrm{E}(S_t/S_0)N((\ln(K/S_0) - m - v^2)/v)$$

We can calculate $\mathrm{PE}(S_t|\ S_t/S_0 < K/S_0) = S_0\ (\mathrm{PE}(S_t/S_0|\ S_t/S_0 < K/S_0))$. This simplifies as follows

$$\begin{aligned} \mathrm{PE}(S_t|\ S_t < K) &= \mathrm{PE}(S_t|\ S_t/S_0 < K/S_0) \\ &= S_0(\mathrm{PE}(S_t/S_0|\ S_t/S_0 < K/S_0)) \\ &= S_0\mathrm{E}(S_t/S_0)N((\ln(K/S_0) - m - v^2)/v) \\ &= S_0 e^{m+1/2v^2}N((\ln(K/S_0) - (\alpha-\delta-1/2\sigma^2)t - \sigma^2 t)/(\sigma\sqrt{t})) \\ &= S_0 e^{(\alpha-\delta)t}N((\ln(K/S_0) - (\alpha-\delta+1/2\sigma^2)t)/(\sigma\sqrt{t})) \\ &= S_0 e^{(\alpha-\delta)t}N(-d_1) \end{aligned}$$

where $d_1 = \frac{\ln(S_0/k)+(\alpha-\delta+1/2\sigma^2)}{\sigma\sqrt{t}}$. Notice that $d_2 = d_1 - \sigma\sqrt{t}$

Since $\mathrm{E}(S_t) = \mathrm{PE}(S_t|\ S_t > K) + \mathrm{PE}(S_t|\ S_t < K)$ then

$$\begin{aligned}
\mathrm{PE}(S_t|\ S_t > K) &= \mathrm{E}(S_t) - \mathrm{PE}(S_t|\ S_t < K) \\
&= S_0 e^{(\alpha-\delta)t} - S_0 e^{(\alpha-\delta)t} N(-d_1) \\
&= S_0 e^{(\alpha-\delta)t}(1 - N(-d_1)) \\
&= S_0 e^{(\alpha-\delta)t} N(d_1)
\end{aligned}$$

Which leads to the following formulas

$$\begin{aligned}
E(S_t|\ S_t < K) &= (S_0 e^{(\alpha-\delta)t} N(-d_1))/N(-d_2) \\
E(S_t|\ S_t > K) &= (S_0 e^{(\alpha-\delta)t} N(d_1))/N(d_2)
\end{aligned}$$

1.4 The Black-Scholes Formula

Substituting in the formulas derived above, we find that for a European call option

$$\begin{aligned}
\mathrm{E(call\ payoff)} &= \Pr(S_t > K)\ [E(S_t \mid S_t > K) - K] \\
&= N(d_2)((S_0 e^{(\alpha-\delta)t} N(d_1))/N(d_2) - K) \\
&= S_0 e^{(\alpha-\delta)t} N(d_1) - K N(d_2) \\
\mathrm{PV_0[E(call\ payoff)]} &= e^{-\alpha t}(S_0 e^{(\alpha-\delta)t} N(d_1) - K N(d_2)) \\
&= S_0 e^{-\delta t} N(d_1) - K e^{-\alpha t} N(d_2)
\end{aligned}$$

So, for a European call option,

$$C = S_0 e^{-\delta t} N(d_1) - K e^{-\alpha t} N(d_2) \tag{1.1}$$

Similarly, substituting in the formulas derived above, we find that, for a European put option

$$\begin{aligned}
\mathrm{E(put\ payoff)} &= \Pr(S_t < K)\ [K - E(S_t \mid S_t < K)] \\
&= N(-d_2)(K - (S_0 e^{(\alpha-\delta)t} N(-d_1))/N(-d_2)) \\
&= K N(-d_2) - S_0 e^{(\alpha-\delta)t} N(-d_1) \\
\mathrm{PV_0[E(call\ payoff)]} &= e^{-\alpha t}(K N(-d_2) - S_0 e^{(\alpha-\delta)t} N(-d_1)) \\
&= K e^{-\alpha t} N(-d_2) - S_0 e^{-\delta t} N(-d_1)
\end{aligned}$$

So, for a European put option,

$$P = Ke^{-\alpha t}N(-d_2) - S_0e^{-\delta t}N(-d_1) \tag{1.2}$$