Geometric Brownian Motion (Lognormal Model)

$$
\begin{aligned}
dS(t) &= (\alpha-\delta)S(t)dt + \sigma S(t)dZ(t) \\
dC(S,t) &= C_s ds + \frac{1}{2}C_{ss}(ds)^2 + C_t dt \\
&= C_s ds[(\alpha-\delta)dt + \frac{1}{2}C_{ss}[(\alpha-\delta)^2 d_t^2 + 2(\alpha-\delta)\sigma dt dZ(t) + (S(t))^2\sigma^2 dZ(t)^2] \\
&\qquad + \sigma dZ(t)]S(t) + C_t dt \\
&= C_s ds(\alpha-\delta)S(t) + \sigma S(t)dZ(t)C_s + \frac{1}{2}\sigma^2 S(t)^2 + C_t dt \\
&= [(\alpha-\delta)S(t)C_s + \frac{1}{2}\sigma^2 S^2 C_{ss}]dt + \sigma S(t)C_s dZ(t) + C_t dt
\end{aligned}
$$

The first line follows by Ito's Lemma.
The second line follows since

$$
\begin{aligned}
dt dZ(t) &= dt Y(t)\sqrt{dt} = dt^{3/2}Y(t) = 0 \\
dt^2 &= 0 \\
dZ(t)^2 &= (dt^{1/2}Y(t))^2 = dt(1) = dt
\end{aligned}
$$

Let $C(S,t) = \ln S(t)$. The partial derivatives can be computed as

$$
\begin{aligned}
C_s &= \frac{\partial C}{\partial S} = \frac{1}{S} \\
C_{ss} &= \frac{\partial^2 C}{\partial S^2} = -\frac{1}{S^2} \\
C_t &= \frac{\partial C}{\partial t} = 0
\end{aligned}
$$

We can plug these into our formula for $dC(S,t)$

$$
\begin{aligned}
dC(S,t) &= [(\alpha-\delta)S\frac{1}{S} + \frac{1}{2}\sigma^2 S^2(-\frac{1}{S^2})]dt + \sigma S\frac{1}{S}dZ(t) + (0)dt \\
dC(S,t) &= (\alpha-\delta)dt + \sigma dZ - \frac{1}{2}\sigma^2 dt \\
d\ln(S(t)) &= (\alpha-\delta-\frac{1}{2}\sigma^2)dt + \sigma dZ \\
\int d\ln S(t) &= \int(\alpha-\delta-\frac{1}{2}\sigma^2)dt + \int \sigma dZ \\
\ln(S(t)) - \ln(S(0)) &= (\alpha-\delta-\frac{1}{2}\sigma^2)\int dt + \sigma\int dz \\
\ln(S(t)) &= \ln(S(0)) + (\alpha-\delta-\frac{1}{2}\sigma^2)t + \sigma Z(t) \\
S(t) &= S(0)e^{(\alpha-\delta-\frac{1}{2}\sigma^2)t+\sigma Z(t)}
\end{aligned}
$$