AMPLITUDE DOMAIN-FEQUENCY REGRESSION

Francisco Parra

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Introduction

The time series can be seen from an aplitude-time domain or an amplitude-frequency domain. The amplitude-frecuency domain are used to analyze properties of filters used to decompose a time series into a trend, seasonal and irregular component investigating the gain function to examine the effect of a filter at a given frequency on the amplitude of a cycle for a particular time series. The ability to decompose data series into different frequencies for separate analysis and later recomposition is the first fundamental concept in the use of spectral techniques in forecasting, such as regression espectrum band, have had little development in econometric work. The low diffusion of this technique has been associated with the computing difficulties caused the need to work with complex numbers, and inverse Fourier transform in order to convert everything back into real terms. But the problems from the use of the complex Fourier transform may be circumvented by carrying out the Fourier transform of the data in real terms, pre-multiplied the time series by the orthogonal matrix Z whose elements are defined in Harvey (1978).

The spectral analysis commences with the assumption that any series can be transformed into a set of sine and cosine waves, and can be used to both identify and quantify apparently nonperiodic short and long cycle processes (first section). In Band spectrum regression (second section) , is a brief summary of the regression of the frequency domain (Engle, 1974) The application of spectral analysis to data containing both seasonal (high frequency) and non-seasonal (low frequency) components may produce adventages, since these different frequencies can be modelled separately and then may be re-combined to produce fitted values. Durbin (1967 and 1969) desing a technique for studying the general nature of the serial dependence in a satacionary time series, that can be use to statistic contraste in This type of exercises (third section). The time-varying regression, or the regression whit the vector of parameters time.varying can be understood in this context (four section).

Spectral analysis

Nerlove (1964) and Granger (1969) were the two foremost researchers on the application of spectral techniques to economic time series.

The use of spectral analysis requires a change of focus from an amplitude-time domain to an amplitude-frequency domain. Thus spectral analysis commences with the assumption that any series, Xt, can be transformed into a set of sine and cosine waves such as:

$$X_t = \eta + \sum_{j=1}^{N}[a_j \cos(2\pi \frac{ft}{n}) + b_j \sin(2\pi \frac{ft}{n})] \tag{1}$$

where η is the mean of the series, a_j and b_j are the amplitude, f is the frequency over a span of n observations, t is a time index ranging from 1 to N where N is the number of periods for which we have observations, the fraction (ft/n) for different values of t converts the discrete time scale of time series into a proportion of 2 and j ranges from 1 to n where n= N/2. The highest observable frequency in the series is n/N (i.e., 0.5 cycles per time interval). High frequency dynamics (large f) are akin to short cycle processes while low frequency dynamics (small f) may be likened to long cycle processes. If we let $\frac{ft}{n} = w$ then equation (1) can be re-written more compactly as:

$$X_t = \eta + \sum_{j=1}^{N}[a_j \cos(\omega_j) + b_j \sin(\omega_j)] \tag{2}$$

Spectral analysis can be used to both identify and quantify apparently non-periodic short and long cycle processes. A given series X_t may contain many cycles of different frequencies and amplitudes and such combinations of frequencies and amplitudes may yield cyclical patterns which appear non-periodic with irregular amplitude. In fact, in such a time series it is clear from equation (2) that each observation can be broken down into component parts of different length cycles which, when added together (along with an error term), comprise the observation (Wilson and Perry, 2004).

The overall effect of the Fourier analysis of N observation to a time date is to partition the variability of the series into components at frequencies $\frac{2\pi}{N}$, $\frac{4\pi}{N}$,...,π.The component at frequency $\omega_p = \frac{2\pi p}{N}$ if called the pth harmonic. For $p \neq \frac{N}{2}$, the equivalent form to write the pth harmonic are:

$$a_p cos\omega_p t + b_p sin\omega_p t = R_p cos(\omega_p t + \phi_p)$$

.

where $R_p = \sqrt{a_p + b_p}$ and $\phi_p = tan^{-1}(\frac{-b_p}{a_p})$

The plot of $I(\omega) = \frac{NR_p^2}{4\pi}$ against ω is called the periodogram of time data. Trend will produce a peak at zero frequency, while seasonal variations produces peaks at the seasonal frquency and at integer multiples of the sesaonal frequency. Then, when a periodogram has a large peak at some frequency ω then related peaks may occurr at 2ω, 3ω,....(Chaftiel, C,2004)

Band spectrum regression

Hannan (1963) first proposed regression analysis in the frequency domain,later examining the use of this technique in estimating distributed lag models (Hannan, 1965, 1967). Engle (1974) demonstrated that regression in the frequency domain has certain advantages over regression in the time domain. Consider the linear regression model

$$y = X\beta + u \tag{3}$$

where X is an n x k matrix of fixed observations on the independent variables, β is a k x I vector of parameters, y is an n x 1 vector of observations on the dependent variable, and u is an n x I vector of disturbance terms each with zero mean and constant variance, σ^2.

The model may be expressed in terms of frequencies by applying a finite Fourier transform to the dependent and independent variables.For Harvey (1978) there are a number of reasons for doing this. One is to permit the application of the technique known as 'band spectrum regression', in which regression is carried out in the frequency domain with certain wavelengths omitted. Another reason for interest in spectral regression is that if the disturbances in (3) are serially correlated, being generated by any stationary stochastic process, then regression in the frequency domain will yield an asymptotically efficient estimator of β.

Engle (1974) compute the full spectrum regression with he complex finite Fourier transform based on the n x n matrix W, in which element (t, s) is given by

$w_{ts} = \frac{1}{\sqrt{n}} e^{i\lambda_t s}$, $s = 0, 1, ..., n-1$

where $\lambda_t = 2\pi \frac{t}{n}$, t=0,1,...,n-1, and $i = \sqrt{-1}$.

Pre-multiplying the observations in observations in (3) by W yields

$$\dot{y} = \dot{X}\beta + \dot{u} \tag{4}$$

where $\dot{y} = Wy$,$\dot{X} = WX$, and $\dot{u} = Wu$.

If the disturbance vector in (4) obeys the classical assumptions, viz. $E[u] = 0$ and $E[uu'] = \sigma^2 I_n$. then the transformed disturbance vector, $\dot{u}$, will have identical properties. This follows because the matrix W is unitary, i.e., $WW^T = I$, where W^T is the transpose of the complex conjugate of W. Furthermore the observations in (4) contain precisely the same amount of information as the untransformed observations in (3).

Application of OLS to (4) yields, in view of the properties of $\dot{u}$, the best linear unbiased estimator (BLUE) of β. This estimator is identical to the OLS estimator in (3), a result which follows directly on taking account of the unitary property of W. When the relationship implied by (4) is only assumed to hold for certain frequencies, band spectrum regression is appropriate, and this may be carried out by omitting the observations in (4) corresponding to the remaining frequencies. Since the variables in (4) are complex, however, Engle (1974) suggests an inverse Fourier transform in order to convert everything back into real terms (Harvey,1974).

The problems which arise from the use of the complex Fourier transform may be circumvented by carrying out the Fourier transform of the data in real terms.

In order to do this the observations in (3) are pre-multiplied by the orthogonal matrix Z whose elements are defined as follows (Harvey,1978):

$$z_{ts} = \begin{cases} \left(\frac{1}{n}\right)^{\frac{-1}{2}} & t = 1 \\ \left(\frac{2}{n}\right)^{\frac{1}{2}} \cos\left[\frac{\pi t(s-1)}{n}\right] & t = 2, 4, 6, ..(n-2) \text{ or } (n-1) \\ \left(\frac{2}{n}\right)^{\frac{1}{2}} \sin\left[\frac{\pi (t-1)(s-1)}{T}\right] & t = 3, 5, 7, .., (n-1) \text{ or } n \\ (n)^{\frac{-1}{2}} (-1)^{s+1} & t = n \text{ if n is even }, s = 1, ...n, \end{cases}$$

The resulting frequency domain regression model is:

$$y^{**} = X^{**}\beta + v \tag{5}$$

where $y^{**} = Zy$,$X^{**} = ZX$ and $v = Zu$.

In view of the orthogonality of Z, $E[vv'] = \sigma^2 I_n$ when $E[uu'] = \sigma^2 I_n$ and the application of OLS to (5) gives the BLUE of β.

Since all the elements of y** and X** are real, model may be treated by a standard regression package. If band spectrum regression is to be carried out, the number of rows in y** and X** is reduced accordingly, and so no problems arise from the use of an inappropriate number of degrees of freedom.

Amplitude domain-frequency regression

Consider now the linear regression model

$$y_t = \beta_t x_t + u_t \tag{6}$$

where x_t is an n x 1 vector of fixed observations on the independent variable, β_t is a n x 1 vector of parameters,y is an n x 1 vector of observations on the dependent variable, and u_t is an n x 1 vector de errores distribuidos con media cero y varianza constante.

Whit the assumption that any series, y_t,x_t,β_t and ut, can be transformed into a set of sine and cosine waves such as:

$$y_t = \eta^y + \sum_{j=1}^{N} [a_j^y \cos(\omega_j) + b_j^y \sin(\omega_j)$$

$$x_t = \eta^x + \sum_{j=1}^{N} [a_j^y \cos(\omega_j) + b_j^y \sin(\omega_j)]$$

$$\beta_t = \eta^\beta + \sum_{j=1}^{N} [a_j^\beta \cos(\omega_j) + b_j^\beta \sin(\omega_j)]$$

Pre-multiplying (6) by Z:

$$\dot{y} = \dot{x}\dot{\beta} + \dot{u}$$

(7)

where $\dot{y} = Zy$,$\dot{x} = Zx$, $\dot{\beta} = Z\beta$ y $\dot{u} = Zu$

The system (7) can be rewritten as (see appendix):

$$\dot{y} = Zx_t I_n Z^T \dot{\beta} + Z I_n Z^T \dot{u}$$

(8)

If we call $\dot{e} = Z I_n Z^T \dot{u}$, It can be found the $\dot{\beta}$ that minimize the sum of squared errors $E_T = Z^T \dot{e}$.

Once you have found the solution to this optimization, the series would be transformed into the time domain.

Seasonal Decomposition by the Fourier Coefficients

The amplitude domain-frequency regression method could be use to decompose a time series into seasonal, trend and irregular components of a time serie y_t of frequency b or number of times in each unit time interval. For example, one could use a value of 7 for frequency when the data are sampled daily, and the natural time period is a week, or 4 and 12 when the data are sampled quarterly and monthly and the natural time period is a year.

If the observation are teken at equal interval of length, $\triangle t$, then the angular frequency is $\omega = frac\pi \triangle t$. The equivalent frequency expressed in cycles per unit time is $f = \frac{\omega}{2\pi} = \frac{1}{2} \triangle t$. Whit only one observation per year, $\omega = \pi$ radians per year or $f = \frac{1}{2}$ cycle per year (1 cicle per two years), variation whit a wavelength of one year has fequency $\omega = 2\pi$ radians per year or $f = 1$ cicle per year.

For example, in a monthly time serie of $N = 100$ observation, the seasonal cycles or the wavelenghth of one year has frequency $f = \frac{100}{12} = 8,33$ cycles for 100 dates. If the time serie are 8 full year, the less seasonal frequency are 1 cycle for year, or 8 cycle for 96 observation. The integer multiplies are $2\frac{N}{12}$,$3\frac{N}{12}$...., and wavelenghth low of one year has frequency are $f < \frac{N}{12}$.

We can use (8) to estimate the fourier coefficient in time serie y_t:

$$\dot{y} = Zt I_n Z^T \dot{\beta} + Z I_n Z^T \dot{u}$$

(9)

being $t = (1, 1, ...1)_N$ or $t = (1, 2, 3, ..., N)_N$.

If $t = (1, 1, 1, ...1)_N$,

$$A = Zt I_n Z^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & 0 & 1 \end{pmatrix}$$

Then

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & 0 & 0 \end{pmatrix}$$

are use in (9) to make the regression band spectrum with the first four coefficient of fourier of the serie $\dot{y}$.

The first$2\frac{N}{12}-1$ rows the A matrix are used to estimate the fourier coefficients corresponding to cycles of low frequency, trend cycles, and rows $2\frac{N}{12}$ and $2\frac{N}{12}+1$ are used to estimate the fourier coefficients of 1 cicle for year. The integer multiplies re the rows $6\frac{N}{12}$, $6\frac{N}{12}+1$, $8\frac{N}{12}$...should be used to obtain the seasonal frequency.

Example:descomponse by amplitude domain-frequency regression. IPI base 2009 in Cantabria

The Industrial Price Index of Cantabria is presented in the table below

The time serie by trend an seasonal is named $TDST$. TD is calculate by band spectrum regresion of the serie y_t and the temporal index t, in which regression is carried out in low amplitude- frequency. The seasonal serie ST result to take away TD to $TDST$, and the irregular serie IR result to take away $TDST$ to y_t (figure 8). The temporal index t used in the exemple are the OLS regression into IPI and the trend index $t = (1, 2, 3, ...N)_N$. The new data fitted are 6 months.

```
> library(descomponer)
> data(ipi)
> descomponer(ipi,12,1,6)
```

```
$data
        y      TDST       TD           ST          IR
1    90.2  98.10295 97.30978   0.79317267  -7.9029508
2    98.8  98.26294 97.35310   0.90984094   0.5370632
3    92.1 100.89796 97.44418   3.45377641  -8.7979604
4   102.7  90.39611 97.57019  -7.17408525  12.3038921
5   107.0 104.21293 97.71583   6.49709735   2.7870737
6    98.3 104.65220 97.86446   6.78773986  -6.3522003
7   100.9  99.70290 97.99941   1.70349678   1.1970962
8    66.3  71.66980 98.10530 -26.43550367  -5.3698002
9   101.4  97.02477 98.16943  -1.14465893   4.3752275
10  111.8 104.26155 98.18289   6.07865327   7.5384530
11  111.4 109.39049 98.14154  11.24895370   2.0095070
12   85.2  96.75562 98.04653  -1.29091634 -11.5556186
13   94.4  97.40501 97.90453  -0.49952276  -3.0050068
14   96.2  96.91526 97.72738  -0.81211509  -0.7152635
```

15	106.5	100.96644	97.53143	3.43500627	5.5335588
16	101.1	90.05066	97.33645	-7.28579348	11.0493435
17	103.5	104.44937	97.16416	7.28521289	-0.9493748
18	99.9	105.44282	97.03668	8.40614645	-5.5428228
19	101.4	99.46136	96.97477	2.48659184	1.9386396
20	58.6	71.70803	96.99624	-25.28820679	-13.1080327
21	99.8	96.04984	97.11446	-1.06462527	3.7501609
22	112.7	102.19482	97.33725	4.85756643	10.5051797
23	103.8	107.12555	97.66613	9.45941951	-3.3255473
24	89.0	94.00170	98.09607	-4.09436928	-5.0016988
25	91.2	97.46393	98.61578	-1.15184687	-6.2639323
26	97.3	99.24689	99.20844	0.03844827	-1.9468916
27	110.2	105.04756	99.85292	5.19463878	5.1524447
28	105.7	98.51657	100.52527	-2.00870413	7.1834300
29	109.9	108.02818	101.20059	6.82759571	1.8718190
30	109.1	107.85836	101.85477	6.00358335	1.2416422
31	104.3	102.81035	102.46642	0.34393236	1.4896472
32	71.9	76.08922	103.01835	-26.92912976	-4.1892175
33	107.1	103.52228	103.49888	0.02340336	3.5777210
34	108.5	110.84951	103.90263	6.94687412	-2.3495083
35	116.6	114.83069	104.23086	10.59982501	1.7693132
36	96.5	99.39304	104.49118	-5.09813349	-2.8930440
37	94.1	103.06947	104.69684	-1.62736211	-8.9694749
38	102.4	104.34413	104.86554	-0.52140933	-1.9441286
39	109.4	110.07981	105.01787	5.06193809	-0.6798122
40	109.0	101.84233	105.17557	-3.33323918	7.1576705
41	113.3	112.03970	105.35963	6.68006277	1.2603027
42	116.5	112.15094	105.58862	6.56232285	4.3490613
43	107.9	107.41395	105.87706	1.53689274	0.4860483
44	76.7	81.77292	106.23434	-24.46141633	-5.0729241
45	111.0	107.63378	106.66395	0.96983306	3.3662191
46	109.3	113.92829	107.16327	6.76501511	-4.6282890
47	119.5	116.32513	107.72393	8.60120036	3.1748653
48	95.1	99.40556	108.33258	-8.92701559	-4.3055647
49	109.6	106.21963	108.97213	-2.75250158	3.3803734
50	109.0	109.16600	109.62330	-0.45729990	-0.1659954
51	125.2	117.08243	110.26630	6.81612951	8.1175682
52	104.8	112.79991	110.88259	1.91731829	-7.9999060
53	123.7	118.04147	111.45638	6.58509579	5.6585264
54	119.7	117.07511	111.97597	5.09914608	2.6248879
55	105.4	112.10797	112.43458	-0.32661704	-6.7079678
56	84.1	87.35960	112.83078	-25.47118373	-3.2595972
57	112.1	114.54919	113.16826	1.38092862	-2.4491852
58	121.6	120.88790	113.45518	7.43271681	0.7121027
59	120.0	122.99230	113.70302	9.28927264	-2.9922964
60	98.6	104.20938	113.92502	-9.71563544	-5.6093825

61	117.6	111.79284	114.13437	-2.34153092	5.8071607
62	117.7	114.72132	114.34239	0.37892915	2.9786770
63	129.7	121.36551	114.55673	6.80877853	8.3344947
64	111.8	114.54055	114.77981	-0.23925106	-2.7405544
65	125.2	120.56175	115.00775	5.55400369	4.6382491
66	121.2	119.68861	115.22976	4.45884810	1.5113907
67	116.8	116.38219	115.42817	0.95402144	0.4178087
68	88.2	93.24652	115.57907	-22.33254875	-5.0465220
69	113.7	118.67409	115.65362	3.02047750	-4.9740947
70	129.0	123.97751	115.61985	8.35766771	5.0224866
71	121.7	122.91386	115.44493	7.46893371	-1.2138609
72	94.4	101.36048	115.09768	-13.73720788	-6.9604758
73	110.3	110.75427	114.55118	-3.79691526	-0.4542684
74	115.3	113.32361	113.78522	-0.46161162	1.9763926
75	112.9	120.91764	112.78847	8.12917310	-8.0176401
76	122.4	115.72389	111.56016	4.16372273	6.6761142
77	116.9	115.85997	110.11114	5.74883452	1.0400257
78	111.2	112.54723	108.46413	4.08309514	-1.3472283
79	115.0	106.38179	106.65331	-0.27151871	8.6182073
80	77.1	82.48854	104.72303	-22.23449808	-5.3885367
81	106.3	105.53320	102.72591	2.80728628	0.7667995
82	115.9	108.27784	100.72029	7.55755537	7.6221576
83	106.7	106.24824	98.76727	7.48097123	0.4517578
84	83.0	82.33012	96.92759	-14.59747024	0.6698806
85	92.2	92.63852	95.25838	-2.61986214	-0.4385154
86	94.3	95.49209	93.81018	1.68191786	-1.1920930
87	96.7	101.06955	92.62431	8.44524186	-4.3695534
88	87.2	93.40126	91.73084	1.67042059	-6.2012597
89	91.0	95.23110	91.14715	4.08395364	-4.2311041
90	91.0	93.29495	90.87735	2.41759705	-2.2949496
91	95.3	91.72655	90.91242	0.81412180	3.5734547
92	70.2	72.07206	91.23112	-19.15905997	-1.8720608
93	98.3	96.61414	91.80156	4.81258031	1.6858600
94	106.9	101.96131	92.58334	9.37796982	4.9386941
95	103.4	99.69841	93.53004	6.16837876	3.7015860
96	86.8	76.60694	94.59195	-17.98501055	10.1930618
97	90.5	91.58205	95.71881	-4.13675936	-1.0820531
98	91.4	96.96138	96.86239	0.09899271	-5.5613815
99	107.7	106.99817	97.97873	9.01943438	0.7018309
100	100.6	103.57335	99.03003	4.54332522	-2.9733542
101	101.9	104.37547	99.98587	4.38960428	-2.4754737
102	105.8	103.81416	100.82400	2.99016015	1.9858428
103	101.5	101.98200	101.53045	0.45155513	-0.4820018
104	75.4	84.47356	102.09916	-17.62560394	-9.0735609
105	101.4	106.70060	102.53117	4.16943068	-5.3006046
106	109.1	110.21571	102.83339	7.38232542	-1.1157133

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107 115.8 108.43199 103.01720   5.41478792   7.3680103
108  98.9  83.96262 103.09698 -19.13435982  14.9373752
109  97.6 100.61337 103.08860  -2.47523096  -3.0133704
110 102.7 106.15524 103.00807   3.14716486  -3.4552370
111 113.2 112.60908 102.87046   9.73861680   0.5909240
112 104.3 104.86051 102.68904   2.17147273  -0.5605105
113 107.6 104.94073 102.47478   2.46594897   2.6592679
114 103.5 102.97034 102.23617   0.73417598   0.5296567
115  97.9 103.13771 101.97924   1.15847160  -5.2377101
116  86.3  86.43621 101.70793 -15.27171205  -0.1362149
117 108.4 107.54872 101.42451   6.12421069   0.8512800
118 103.5 110.78491 101.13015   9.65475735  -7.2849120
119 103.5 105.63306 100.82547   4.80758534  -2.1330585
120  89.0  79.26858 100.51100 -21.24242208   9.7314196
121  94.5  96.46301 100.18757  -3.72456119  -1.9630133
122  97.7 101.07087  99.85653   1.21434090  -3.3708709
123 112.9 108.95707  99.51977   9.43730023   3.9429308
124  97.6 102.29885  99.17964   3.11920139  -4.6988451
125 111.6 101.49461  98.83873   2.65588053  10.1053939
126 103.8 100.38433  98.49949   1.88483843   3.4156730
127  97.3  99.86119  98.16396   1.69722583  -2.5611893
128  86.6  85.59797  97.83342 -12.23544906   1.0020291
129  94.7 102.83054  97.50813   5.32241116  -8.1305408
130 100.3 104.16297  97.18725   6.97572081  -3.8629726
131  95.4 100.23458  96.86887   3.36570557  -4.8345776
132  85.4  73.81507  96.55020 -22.73512912  11.5849294
133  96.3  94.28154  96.22792  -1.94637911   2.0184633
134  94.5 100.44664  95.89864   4.54799625  -5.9466357
135  98.1 106.05061  95.55947  10.49114763  -7.9506146
136 105.0  96.38455  95.20852   1.17602640   8.6154493
137 101.0  95.73162  94.84547   0.88615421   5.2683754
138  98.8  94.09821  94.47189  -0.37367330   4.7017869
139  91.5  96.06831  94.09150   1.97681574  -4.5683142
140  80.5  82.65898  93.71019 -11.05121750  -2.1589750
141  94.6 100.14628  93.33580   6.81048107  -5.5462812
142 100.6 102.10782  92.97766   9.13016622  -1.5078238
143  91.8  96.13222  92.64596   3.48625204  -4.3322152
144  82.1  69.11409  92.35098 -23.23689059  12.9859105
145  91.8  89.46843  92.10215  -2.63372390   2.3315733
146  92.6  94.67246  91.90720   2.76525664  -2.0724572
147 100.1 101.14934  91.77132   9.37802112  -1.0493402
148  95.4  91.86183  91.69650   0.16533569   3.5381669

$fitted
  TDST_fitted TD_fitted  ST_fitted
1    92.42839  91.68110  0.7472933
```

2	92.54072	91.68386	0.8568575
3	94.98286	91.73156	3.2513002
4	85.06135	91.81205	-6.7507037
5	98.02203	91.91090	6.1111293
6	98.39431	92.01245	6.3818529

Bibliography

Chatfield, Cris (2004). "The Analysis of Time Series: An Introduction (6th edn.)", 2004. CRC Press

Engle, Robert F. (1974), "Band Spectrum Regression", International Economic Review 15,1-11.

Hannan, E.J. (1963), "Regression for Time Series", in Rosenblatt, M. (ed.), Time Series Analysis, New York, John Wiley.

Harvey, A.C. (1978), "Linear Regression in the Frequency Domain", International Economic Review, 19, 507-512.

Wilson, P.J. and Perry, L.J. (2004). "Forecasting Australian Unemployment Rates Using Spectral Analysis" Australian Jurnal of Labour Economics, vol 7,no 4, December 2004, pp 459-480.

Appendix

The multiplication of two harmonic series of diferent frequency:

$$[a_j \cos(\omega_j) + b_j \sin(\omega_j)]x[a_i \cos(\omega_i) + b_i \sin(\omega_i)]$$

gives the following sum:

$$a_j a_i \cos(\omega_j) \cos(\ omega_i) + a_j b_i \cos(\omega_j) \sin(\omega_i)$$

$$+a_i b_j \sin(\omega_j) \cos(\omega_i) b_i \sin(\omega_i) + b_j b_i \sin(\omega_j) \sin(\omega_i)$$

that using the identity of the products of sines and cosines gives the following results:

$$\frac{a_j a_i + b_j b_i}{2} \cos(\omega_j - \omega_i) + \frac{b_j a_i - b_j a_i}{2} \sin(\omega_j - \omega_i)$$

$$+\frac{a_j a_i - b_j b_i}{2} \cos(\omega_j + \omega_i) + + \frac{b_j a_i + b_j a_i}{2} \sin(\omega_j + \omega_i)$$

The circularity of ω determines that the product of two harmonics series resulting in a new series in which the Fourier coefficients it's a linear combination of the Fourier coefficients of the two harmonics series.

In the following two series:

$$y_t = \eta^y + a_0^y \cos(\omega_0) + b_0^y \sin(\omega_0) + a_1^y \cos(\omega_1) + b_1^y \sin(\omega_1) + a_2^y \cos(\omega_2) + b_2^y \sin(\omega_2) + a_3^y \cos(\omega_3)$$

$$x_t = \eta^x + a_0^x \cos(\omega_0) + b_0^x \sin(\omega_0) + a_1^x \cos(\omega_1) + b_1^x \sin(\omega_1) + a_2^x \cos(\omega_2) + b_2^x \sin(\omega_2) + a_3^x \cos(\omega_3)$$

given a matrix $\Theta^{\dot{x}\dot{x}}$ of size 8x8 :

$$\Theta^{\dot{x}\dot{x}} = \eta^x I_8 + \frac{1}{2} \begin{pmatrix} 0 & a_0^x & b_0^x & a_1^x & b_1^x & a_2^x & b_2^x & 2a_3^x \\ 2a_0^x & a_1^x & b_1^x & a_0^x + a_2^x & b_0^x + b_2^x & a_1^x + 2a_3^x & b_1^x & 2a_2^x \\ 2b_0^x & b_1^x & -a_1^x & -b_0^x + b_2^x & a_0^x - a_2^x & -b_1^x & a_1^x - a_3^x & -2b_2^x \\ 2a_1^x & a_0^x + a_2^x & -b_0^x + b_2^x & 2a_3^x & 0 & a_0^x + a_2^x & b_0^x - b_2^x & 2a_1^x \\ 2b_1^x & a_0^x + b_2^x & -b_0^x - a_2^x & 0 & -2a_3^x & -b_0^x + b_2^x & a_0^x - a_2^x & -2b_1^x \\ 2a_2^x & a_1^x + 2a_3^x & -b_1^x & a_0^x + a_2^x & -b_0^x - b_2^x & a_1^x & -b_1^x & 2a_0^x \\ 2b_2^x & b_1^x & a_1^x - 2a_3^x & b_0^x - b_2^x & a_0^x - a_2^x & -b_1^x & -a_1^x & -2b_0^x \\ 2a_3^x & a_2^x & -b_2^x & a_1^x & -b_1^x & a_0^x & -b_0^x & 0 \end{pmatrix}$$

Demonstrates that:

$$\dot{z} = \Theta^{\dot{x}\dot{x}} \dot{y}$$

where $\dot{y} = Wy$,$\dot{x} = Wx$, and $\dot{z} = Wz$.

$$z_t = x_t y_t = W^T \dot{x} W^T \dot{y} = W^T W x_t W^T \dot{y} = x_t I_n W^T \dot{y}$$

$$W^T \dot{z} = x_t I_n W^T \dot{y}$$

$$\dot{z} = W^T x_t I_n W \dot{y}$$

It is true that;

$$x_t I_n = W^T \Theta^{\dot{x}\dot{x}} W$$

and

$$\Theta^{\dot{x}\dot{x}} = W^T x_t I_n W$$