



bartcs: Bayesian Additive Regression Trees for Confounder Selection in R

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Abstract

This article presents an overview of the **bartcs** R package, which employs a Bayesian additive regression tree-based method for selecting confounders. It uses a Dirichlet distribution as a common variable selection probability prior, updating both the exposure and outcome models simultaneously while fitting BART priors for each. This data-driven method determines which variables (i.e., confounders) affect both models by assigning more posterior weight to them. It supports continuous and binary exposure variables, as well as continuous outcome variables, and is written in C++ for improved computational speed. Additionally, it can take advantage of multiple threads for parallel computing if OpenMP is available on the platform.

Keywords: Bayesian nonparametric, causal inference, high-dimensional confounders, continuous outcome.

1. Introduction: Confounder selection in R

In observational studies, drawing causality always relies on the ignorability assumption (Rosenbaum and Rubin 1983b) that all confounders are included in the adjustment procedure. In many recent applications, the number of potential confounders is often enormous, making it difficult to select the optimal set of true confounders among them. In this context, the optimal set is a confounder set with an appropriate level of uncertainty that reduces bias in estimating the final causal effect.

The main distinction between confounder selection and the traditional variable selection method is that variables that affect both exposure and outcome should be chosen. Several criteria need to be met by the selected confounder in order to reduce the bias of estimated causal effects. Among them, “disjunctive cause criterion” (VanderWeele 2019) requires that the chosen variable be related to exposure and/or outcome. A better condition than this

is “disjunctive cause criterion without instruments”(VanderWeele 2019), which removes the variables related to exposure but not directly associated with outcome. Manually identifying a set of confounders that meet these criteria among a large number of potential confounders is challenging.

Methods based on data and statistical models for performing such confounder selection have recently been proposed. One such method is the Bayesian adjustment for confounding (BAC) proposed by Wang, Parmigiani, and Dominici (2012); Lefebvre, Delaney, and McClelland (2014), which connects exposure and outcome models through common variable inclusion indicator variables to identify confounders. Wang, Dominici, Parmigiani, and Zigler (2015)) later modified the BAC method to work with generalized linear outcome models. Wilson and Reich (2014) suggested a method based on decision theory with a similar goal, which performs well for a variety of sample sizes. In terms of selecting relevant covariates for use in propensity score, Shortreed and Ertefaie (2017) proposed the outcome-adaptive LASSO method. In addition, Häggström (2018) proposed a method for identifying the causal structure and estimating the causal effect using a probability graphical model.

Despite the advantages of the previously mentioned methods, they each have limitations as outlined in Table 1. To address these shortcomings, Kim, Tec, and Zigler (2023) proposed a novel Bayesian non-parametric model that aims to overcome these limitations. This suggested a new method that employs Bayesian additive regression trees (BART; Chipman, George, McCulloch *et al.* (2010)) with a shared prior for the selection probabilities, which links the exposure and outcome models. This approach allows for the flexibility and precision of a Bayesian nonparametric model, while also identifying and integrating covariates that are related to both the exposure and outcome into the final model. This paper introduces **bartcs**, a new R (R Core Team 2021) package developed by Yoo (2022) that implements the Bayesian additive regression trees method for confounder selection proposed by Kim *et al.* (2023). The package, which is written in C++ and integrated into R via Rcpp for fast computation and easy use, can be downloaded from the Comprehensive R Archive Network (CRAN) at <https://cran.r-project.org/package=bartcs>.

In this paper, we provide an overview of the package, including installation instructions, usage examples, and a demonstration of its performance on simulated data. We also include a comparison with other existing confounder selection methods. Our aim is to provide researchers with a useful tool for identifying relevant covariates in their causal inference studies and to enable them to make more accurate causal inferences.

2. Overview of Model

We first express causal estimation within a potential outcome framework (Rubin 1974). For each unit $i = 1, \dots, N$, the potential outcome for the i -th unit is defined as $Y_i(a)$, representing the potential value of the outcome Y_i that could be observed under the exposure $A = a$. The target causal estimand is

$$\Delta(a, a') = E[Y(a) - Y(a')],$$

which represents the average difference between two potential outcomes under two different exposure levels a and a' . However, unlike randomized trials, the exposure assignment is not randomized in observational studies, making it impossible to directly identify either $E[Y(a)]$ or $E[Y(a')]$ from observed data. With a proper set of confounders \mathbf{X}_i , the following strong

Package	Prog. Lang.	Description
bacr (Wang <i>et al.</i> 2012, 2015)	R	Assume (generalized-) linear models for exposure and outcome. Supports binomial, Poisson, Gaussian exposure and outcome.
BayesPen (Wilson and Reich 2014)	R	Assume linear models for exposure and outcome. Support continuous outcome.
CovSelHigh (Häggström 2017)	R	Confounder selection performed via either Markov/Bayesian networks (Model-free selection of confounders)
BART [†] (Sparapani, Spanbauer, and McCulloch 2021)	C++	Incorporate the Dirichlet sparse prior of Linero (2018) for variable selection in the BART outcome model. Support continuous outcome.
bartcs (Yoo 2022)	C++	Use BART outcome and exposure models with the common Dirichlet prior for variable selection. Support binary and continuous exposure, and continuous outcome.

Table 1: Summary of different confounder selection models. †Note that this model (DBART) is not primarily focused on confounder selection, but rather variable selection, and this variable selection functionality is enabled by setting `sparse=TRUE` in `wbart` function.

ignorable treatment assignment assumption (Rosenbaum and Rubin 1983a) holds

$$Y_i(a) \perp A_i | \mathbf{X}_i,$$

and $0 < Pr(A_i = 1 | \mathbf{X}_i = \mathbf{x}) < 1$ for all $\mathbf{x}; i = 1, \dots, N$. With this assumption in place, we can represent the causal effect by the following equation of the observable quantities:

$$\Delta(a, a'; \mathbf{x}) = E[Y | A = a, \mathbf{X} = \mathbf{x}] - E[Y | A = a', \mathbf{X} = \mathbf{x}],$$

and finally identify and estimate the target estimand $\Delta(a, a')$ by averaging over confounders \mathbf{X} . Thus, the two key tasks in estimating causal effects are identifying the appropriate confounders among a potentially large set of covariates, and determining the outcome model (i.e., $E[Y | A = a, \mathbf{X} = \mathbf{x}]$) with flexibility and precision. The **bartcs** R package was developed to address these challenges by utilizing Bayesian Additive Regression Trees (BART) models for confounder selection and causal effect estimation.

2.1. Overview of BART

The BART model (Chipman *et al.* 2010) is an ensemble of decision trees that can be represented by the equation:

$$y_i = f(\mathbf{X}_i) + \epsilon_i \approx \sum_{t=1}^T g(\mathbf{X}_i; \mathcal{T}_t, \mathcal{M}_t) + \epsilon_i,$$

where ϵ_i follows a normal distribution with mean 0 and variance σ^2 , and $g(\mathbf{X}_i; \mathcal{T}_t, \mathcal{M}_t)$ is a function that maps the tree structure and parameters to the response, for all $i =$

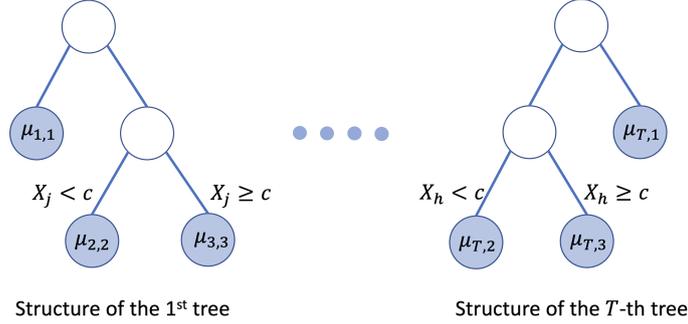


Figure 1: The tree structures consist of T trees, each with a node represented by a circle. Terminal nodes, shown in blue, have μ values. The outcome estimate \hat{Y} of each observation is calculated by adding up the μ values of the terminal nodes where the observation falls within each tree. The method used to split each internal node into two different children node is the “splitting rule,” which consists of a “splitting variable” (i.e., X_j) and a “splitting value” (i.e., c).

$1, \dots, N$. For each of T distinct trees, \mathcal{T}_t represents the structure of the t -th tree and $\mathcal{M}_t = \{\mu_{t,1}, \mu_{t,2}, \dots, \mu_{t,n_t}\}$ represents its mean parameters at the terminal nodes. Each tree has internal nodes that are split based on a “splitting variable” X_j and “splitting value” c (Figure 1).

In the Markov Chain Monte Carlo (MCMC) update, Bayesian backfitting (Hastie, Tibshirani *et al.* 2000) is utilized within a Metropolis-within-Gibbs sampler. This involves fitting each tree in the ensemble sequentially, using the residual responses: $\mathbf{R}_{-t} := \mathbf{y} - \sum_{j \neq t} g(\mathbf{X}; \mathcal{T}_j, \mathcal{M}_j)$ where \mathbf{R}_{-t} denotes unexplained outcome residuals for the t -th tree. In each iteration of the MCMC update, a new tree structure is proposed by randomly selecting one of three possible tree alterations:

GROW: Choose a terminal node at random, and create two new terminal nodes. This process involves randomly selecting a predictor, X_j , and its associated “splitting value,” c , to create the two new terminal nodes.

PRUNE: Pick an internal node at random where both children are terminal nodes (known as a “singly internal node” (Kapelner and Bleich 2016)) and remove both of its children (thus making it a terminal node).

CHANGE: Select an internal node at random and modify its splitting variable and value according to the priors.

Specifically, when using the grow and change alterations, a new covariate is randomly selected from a set of q available covariates as the splitting variable, according to the assumed priors. The original BART model used a uniform prior of $\{1/q, 1/q, \dots, 1/q\}$ on the selection probabilities $s = (s_1, s_2, \dots, s_q)$. However, to promote sparsity, Linero (2018) proposed using a Dirichlet prior $(s_1, s_2, \dots, s_q) \sim \mathcal{D}(\alpha/q, \dots, \alpha/q)$. For a detailed explanation of the parameter setting and the steps involved in computing the posterior, refer to Kim *et al.* (2023).

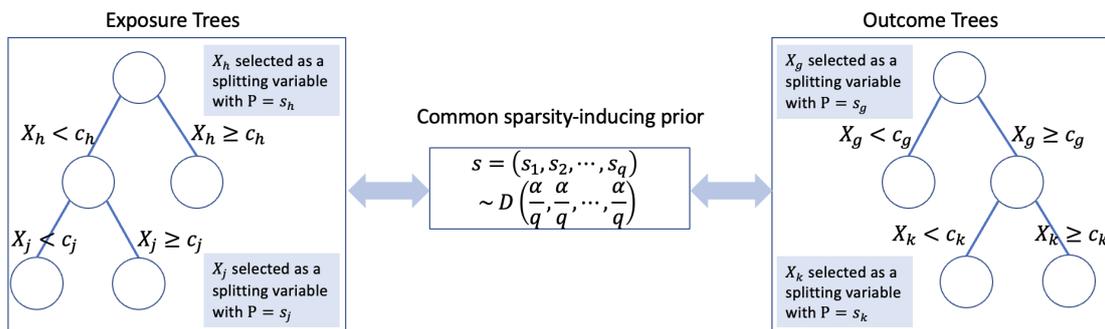


Figure 2: A shared sparsity-inducing prior for the selection probability vector connects the exposure model and outcome model, enabling the selection of the splitting variable. The selection probability vector is updated based on the number of splitting variables used to describe each tree.

2.2. BART confounder selection

The `bartcs` package in R is designed for selecting confounding variables, particularly when a large number of potential confounding variables are present, and for estimating the average treatment effect (ATE) given the chosen set of confounding variables. To accomplish this, the package uses the Bayesian Additive Regression Trees (BART) model to specify the exposure and outcome as follows:

$$P(A_i = 1) = \Phi \left(\sum_{t=1}^T g_a(\mathbf{X}_i; \mathcal{T}_t, \mathcal{M}_t) \right) \quad (1)$$

$$Y_i = \sum_{t=1}^T g_y(A_i, \mathbf{X}_i; \mathcal{T}'_t, \mathcal{M}'_t) + \epsilon_i, \quad (2)$$

where $\epsilon_i \sim N(0, \sigma^2)$ for $i = 1, \dots, N$. In Eq. (1), $\Phi(\cdot)$ is the standard normal cumulative distribution function. Note that it is required to replace Eq. (1) with $A_i = \sum_{t=1}^T g_a(\mathbf{X}_i; \mathcal{T}_t^*, \mathcal{M}_t^*) + \epsilon_i^*$, $\epsilon_i^* \sim N(0, \tau^2)$ when considering a continuous exposure. We incorporate a common sparsity-inducing Dirichlet prior $(s_1, s_2, \dots, s_q) \sim \mathcal{D}(\alpha/q, \dots, \alpha/q)$ on Eq. (1) and (2) resulting in a conjugate update $(s_1, s_2, \dots, s_q) \sim \mathcal{D}(\alpha/q + n_1^a + n_1^y, \dots, \alpha/q + n_q^a + n_q^y)$ where n_j^a and n_j^y are the number of splits on potential confounder X_j in Eq. (1) and Eq. (2), respectively (Figure 2).

If a particular covariate, X_j , is frequently used as a splitting variable in either the model for A or the model for Y , the model will assign more weight to the selection probability s_j through larger values of n_j^a or n_j^y . This means that the selection probabilities will tend to favor covariates that have a relationship with A , Y , or both A and Y . The final confounders chosen for effect estimation in the model for Y will be those that were proposed for splitting through this prior and were accepted during the updating step of the model for Y , which will further prioritize variables that have a relationship with Y . This characteristic satisfies the “disjunctive cause criterion without instruments” in confounder selection. Please see [Kim et al. \(2023\)](#) for further discussion on this prioritization.

Separate outcome models

For a binary exposure, we separate the outcome model in Eq (2) into two distinct models, in order to align the dimensions of the covariates in both the exposure and outcome models (note that Eq (2) includes exposure A_i as an additional covariate). For each $A = a \in \{0, 1\}$,

$$Y_i = \sum_{t=1}^T g_y^a(\mathbf{X}_i; \mathcal{T}_t^a, \mathcal{M}_t^a) + \epsilon_i^a, \quad \epsilon_i^a \sim N(0, \sigma_a^2), \quad (3)$$

for $i \in N_a$ where N_a denotes a set of units under each exposure arm $a \in \{0, 1\}$. A sparsity-inducing prior is applied to (s_1, s_2, \dots, s_q) , which is shared among three models: one for exposure and two for outcomes. The resulting update based on this prior is $(s_1, s_2, \dots, s_q) \sim \mathcal{D}(\alpha/q + n_1^a + n_1^{y_1} + n_1^{y_0}, \dots, \alpha/q + n_q^a + n_q^{y_1} + n_q^{y_0})$, where $n_j^{y_1}$ and $n_j^{y_0}$ represent the number of splits on the confounder X_j in two separate outcome models.

Single outcome model

Using two separate outcome models for two exposure levels, as outlined in Hill (2011) and Hahn, Murray, and Carvalho (2020), can result in biased estimates if there is a lack of common support. While a single outcome model can be a viable alternative, it can be challenging to apply a shared sparsity-inducing prior to (s_1, s_2, \dots, s_q) due to differences in covariate dimensions between the exposure and outcome models. Let $s = (s_0, s_1, s_2, \dots, s_q)$ represent the selection probabilities, with s_0 denoting the probability of exposure A in the outcome model. To apply this vector to the exposure model, s is transformed to $s' = (s_1/(1 - s_0), s_2/(1 - s_0), \dots, s_q/(1 - s_0))$. Then, updating s is based on the equation (likelihood \times prior):

$$Q = \left(\frac{1}{1 - s_0} \right)^{\sum_{j=1}^q n_j^a} s_0^{n_0^y + \alpha/q - 1} s_1^{n_1^y + n_1^a + \alpha/q - 1} \dots s_q^{n_q^y + n_q^a + \alpha/q - 1},$$

using the Metropolis-Hastings algorithm. The proposal distribution for s is designed to follow the full conditional in the separate outcome model, $\mathcal{D}(n_0^y + c + \alpha/q, n_1^a + n_1^y + \alpha/q, n_2^a + n_2^y + \alpha/q, \dots, n_q^a + n_q^y + \alpha/q)$, and a positive value c is added to prevent proposals for infrequent exposure. For a detailed explanation of the posterior computation step, refer to the appendix and Kim *et al.* (2023).

Given the M set of posterior samples for BART parameters, the causal effect estimand $\Delta(a, a')$ can be estimated using either the separate model or the single model. For the separate outcome model, the estimate is given by

$$\hat{\Delta}(1, 0) = \frac{1}{N} \sum_{i=1}^N \left[\frac{1}{M} \sum_{m=1}^M \left\{ \sum_{t=1}^T g_y^{1,(m)}(\mathbf{X}_i; \mathcal{T}_t^1, \mathcal{M}_t^1) - \sum_{t=1}^T g_y^{0,(m)}(\mathbf{X}_i; \mathcal{T}_t^0, \mathcal{M}_t^0) \right\} \right],$$

where $g_y^{a,(m)}$ is the m -th posterior samples for $A = a \in 0, 1$. For the single outcome model, the estimate is given by

$$\hat{\Delta}(1, 0) = \frac{1}{N} \sum_{i=1}^N \left[\frac{1}{M} \sum_{m=1}^M \left\{ \sum_{t=1}^T g_y^{(m)}(1, \mathbf{X}_i; \mathcal{T}_t, \mathcal{M}_t) - \sum_{t=1}^T g_y^{(m)}(0, \mathbf{X}_i; \mathcal{T}_t, \mathcal{M}_t) \right\} \right],$$

where $g_y^{(m)}$ is the m -th posterior samples.

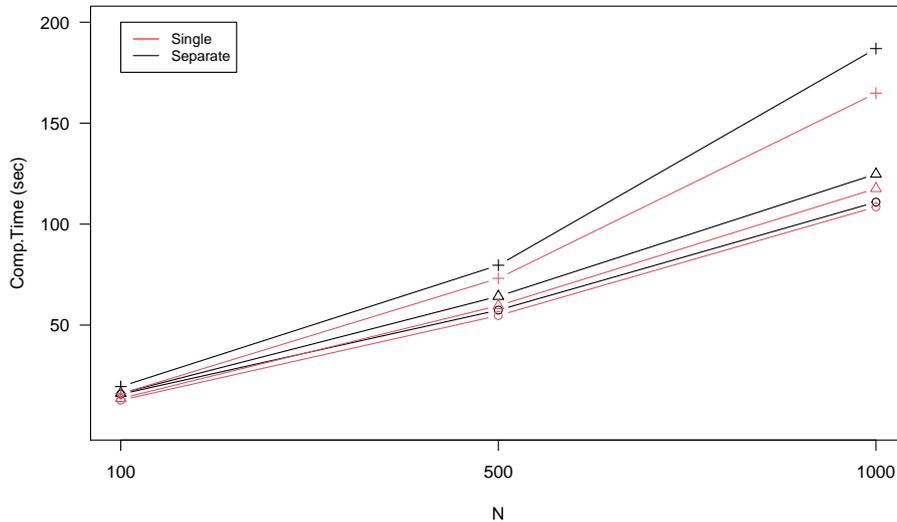


Figure 3: The computation times for both the single outcome model (red) and separate outcome model (black) based on the number of observations (N). A cross symbol (+) represents the scenario where the number of potential confounders (P) is equal to the number of observations (N), a triangle (Δ) represents the scenario where $P = N \times 0.5$ and a circle (\circ) represents the scenario where $P = N \times 0.3$. These results are obtained from 20000 MCMC iterations.

3. Preliminaries

The **bartcs** R package makes it easy to implement the confounder selection process described in the previous section. It includes two main functions, `separate_bart()` for the separate outcome model and `single_bart()` for the single outcome model. Along with the summary of the estimated causal effects, the package provides convergence diagnostic information (Gelman-Rubin diagnostic, [Gelman and Rubin \(1992\)](#)) and plots of posterior inclusion probabilities and convergence.

bartcs offers multi-threading support through Open Multi-Processing (OpenMP), an API for shared memory parallel programming that manages thread creation, management, and synchronization for efficient data and computation division among different threads. This allows **bartcs** to specify intensive computations as parallel regions, leading to improved computational efficiency through parallel computing.

Figure 3 shows the computational speed of two models, the separate and single models. The number of observations (N) was evaluated with 100, 500, and 1000, while the number of potential confounders (P) was tested in three different situations, $N \times 0.3$ (circle), $N \times 0.5$ (triangle), and $N \times 1$ (cross). The speed was calculated based on 20000 MCMC iterations. The separate model had a computational speed ranging from 16 to 187 seconds, while the single model had a slight decrease in speed with a minimum of 13 seconds and a maximum of 165 seconds, depending on the combination of (N, P). Both models were considered to have similar computational speed, given the 20000 MCMC iterations. The single model was found

to be more efficient as it fits two BART models (exposure and one outcome model). Given that the single model has slightly smaller biases and MSEs in many different scenarios in [Kim et al. \(2023\)](#), it is recommended to use the single model (`single_bart()` function) due to its faster computation speed if N is large.

The package **bartcs** is available under the general public license (GPL ≥ 3) from the Comprehensive R Archive Network (CRAN) at <https://cran.r-project.org/package=bartcs> and can be installed and loaded into the current R session as follows:

```
R> install.packages("bartcs")
R> library("bartcs")
```

In the following sections, we will showcase the practical usage of the features in the **bartcs** package using simulated examples and IHDP data.

4. Simulated Example

As a simple example of the **bartcs** package, we use a simulated dataset from Scenario 1 in [Kim et al. \(2023\)](#) to illustrate its features. The data-generating model incorporates both the non-linear propensity score and outcome models, and serves to evaluate the ability to detect 5 true confounding variables out of a huge set of possibilities, along with the precision of the model's estimation. The dataset consists of 300 observations with 100 potential confounders ($X_1 - X_{100}$), each generated from a normal distribution with mean 0 and variance 1. Of the 100 possible confounders, $X_1 - X_5$ are true confounders. The outcome model includes the five true confounders and two additional predictors, X_6 and X_7 as follows:

$$\begin{aligned} P(A_i = 1) &= \Phi(0.5 + 0.5h_1(X_{i,1}) + 0.5h_2(X_{i,2}) - 0.5|X_{i,3} - 1| + 1.5X_{i,4}X_{i,5}) \\ Y_i &\sim N(\mu(\mathbf{X}_i), 0.3^2) \\ \mu(\mathbf{X}_i) &= h_1(X_{i,1}) + 1.5h_2(X_{i,2}) - A_i + 2|X_{i,3} + 1| + 2X_{i,4} + \exp(0.5X_{i,5}) \\ &\quad - 0.5A_i|X_{i,6}| - A_i|X_{i,7} + 1| \end{aligned}$$

where $h_1(x) = (-1)^{I(x < 0)}$ and $h_2(x) = (-1)^{I(x \geq 0)}$ for $i = 1, \dots, 300$. The data was generated with the following code:

```
R> set.seed(42)
R> N <- 300
R> P <- 100
R> cov <- list()
R> for(i in 1:P) {
+   cov[[i]] <- rnorm(N, 0, 1)
+ }
R> X <- do.call(cbind, cov)
R> h1 <- ifelse(X[, 1] < 0, 1, -1)
R> h2 <- ifelse(X[, 2] < 0, -1, 1)
R> prob <- pnorm(0.5 + h1 + h2 - 0.5 * abs(X[, 3] - 1) +
+           1.5 * X[, 4] * X[, 5])
R> Trt <- rbinom(N, 1, prob)
```

```
R> mu1 <- 1 * h1 + 1.5 * h2 - 1 + 2 * abs(X[, 3] + 1) +
+ 2 * X[, 4] + exp(0.5 * X[, 5]) -
+ 0.5 * 1 * abs(X[, 6]) - 1 * 1 * abs(X[, 7] + 1)
R> mu0 <- 1 * h1 + 1.5 * h2 - 0 + 2 * abs(X[, 3] + 1) +
+ 2 * X[, 4] + exp(0.5 * X[, 5]) -
+ 0.5 * 0 * abs(X[, 6]) - 1 * 0 * abs(X[, 7] + 1)
R> Y1 <- rnorm(N, mu1, 0.3)
R> Y0 <- rnorm(N, mu0, 0.3)
R> Y <- Trt * Y1 + (1 - Trt) * Y0
```

With a generated data set, we fit the BART confounder selection model (the separate outcome model) using `separate_bart()`.

```
library(bartcs)
R> separate_fit <- separate_bart(
+ Y = Y, trt = Trt, X = X, num_tree = 50, num_chain = 4,
+ num_burn_in = 10000, num_thin = 10, num_post_sample = 1000
+ )
```

The following are the main arguments used in the `separate_bart()` function call:

- `Y` represents a vector of observed outcome values.
- `trt` represents a vector of exposure(treatment) values, which can be either binary or continuous depending on the function. Binary treatment values need to be either 0 or 1.
- `X` is a data frame of potential confounders.

The following are the remaining settings for the fit: 4 MCMC chains (`num_chain`) with 50 trees (`num_tree`) are used. Each MCMC chain involves 20000 iterations, with 10000 burn-in iterations (`num_burn_in`) and a thinning factor of 10 (`num_thin`). There are other optional arguments available for hyper-parameter settings with the following default values:

- $\alpha = 0.95$ (`alpha`) and $\beta = 2$ (`beta`): these govern the probability that a node at depth d is nonterminal as follows

$$\alpha(1 + d)^{-\beta}.$$

- $\nu = 3$ (`nu`) and $q = 0.95$ (`q`): to set a conjugate prior for the variance σ^2 with $\sigma^2 \sim \nu\lambda/\chi_\nu^2$, we use the following equation to determine the values $P(\sigma < \hat{\sigma}) = q$, where $\hat{\sigma}$ represents the residual standard deviation obtained from a linear regression of Y on X .
- $P_{\text{GROW}} = 0.28$, $P_{\text{PRUNE}} = 0.28$, $P_{\text{CHANGE}} = 0.44$: probabilities of three tree alteration steps.
- `dir_alpha = 5`: this is an initial value for hyperparameter α in the sparsity inducing Dirichlet prior $\mathcal{D}(\alpha/q, \alpha/q, \dots, \alpha/q)$.

```
R> separate_fit
```

```
`bartcs` fit by `separate_bart()``
```

	mean	2.5%	97.5%
ATE	-2.3528698	-2.609140	-2.0964852
Y1	0.7147532	0.504403	0.9259257
Y0	3.0676229	2.927474	3.2089274

The `separate_bart()` returns a S3 `bartcs` object. A `bartcs` object displays the posterior means and 95% credible intervals for the sample average treatment effect (ATE), and the potential outcomes $Y(1)$ and $Y(0)$. It is important to note that the true values for the ATE , $E[Y(1)]$, and $E[Y(0)]$ are -2.55 , 0.64 , and 3.19 respectively, and the 95% credible intervals produced by the `separate_bart()` function include these values.

For a more in-depth understanding of the output, the `summary()` function can be used. It provides information on the treatment values, tree structure, MCMC chain, gelman-rubin diagnostic, and outcomes for each of the MCMC chains.

```
R> summary(separate_fit)
```

```
`bartcs` fit by `separate_bart()``
```

Treatment Value

```
Treated group : 1
Control group : 0
```

Tree Parameters

Number of Tree	: 50	Value of alpha	: 0.95
Prob. of Grow	: 0.28	Value of beta	: 2
Prob. of Prune	: 0.28	Value of nu	: 3
Prob. of Change	: 0.44	Value of q	: 0.95

Chain Parameters

Number of Chains	: 4	Number of burn-in	: 10000
Number of Iter	: 20000	Number of thinning	: 10
Number of Sample	: 1000		

Outcome Diagnostics

```
Gelman-Rubin : 0.9990445
```

Outcome

estimand	chain	2.5%	1Q	mean	median	3Q	97.5%
ATE	1	-2.6001705	-2.4340846	-2.3443589	-2.3476330	-2.2551920	-2.0799121
ATE	2	-2.6087384	-2.4259789	-2.3397221	-2.3408950	-2.2453614	-2.0859052
ATE	3	-2.6144787	-2.4517900	-2.3631684	-2.3611169	-2.2735971	-2.1222551
ATE	4	-2.6132885	-2.4442093	-2.3642297	-2.3679466	-2.2784649	-2.1087727
ATE	agg	-2.6091399	-2.4403736	-2.3528698	-2.3531103	-2.2646584	-2.0964852
Y1	1	0.5252200	0.6459721	0.7246384	0.7213282	0.7977788	0.9508965

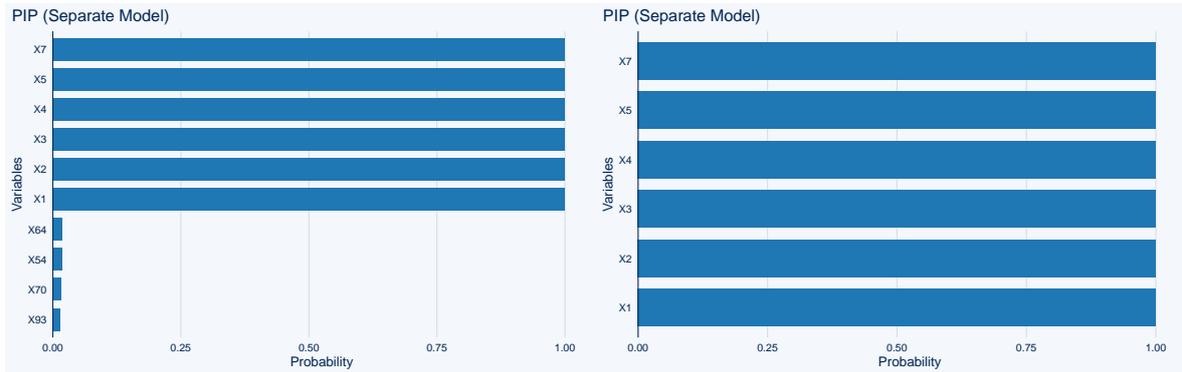
(a) Plotting PIP with `top_n` argument(b) Plotting PIP with `threshold` argument

Figure 4: Posterior inclusion probability (PIP) plots

Y1	2	0.5047637	0.6539640	0.7265392	0.7287699	0.8065149	0.9362558
Y1	3	0.4895294	0.6352240	0.7052498	0.7095617	0.7780091	0.9125417
Y1	4	0.4997177	0.6309322	0.7025852	0.6981323	0.7700263	0.9085243
Y1	agg	0.5044030	0.6412497	0.7147532	0.7146822	0.7897952	0.9259257
Y0	1	2.9274229	3.0204668	3.0689973	3.0659627	3.1151496	3.2107029
Y0	2	2.9165799	3.0225901	3.0662613	3.0653160	3.1149797	3.2019031
Y0	3	2.9348665	3.0156523	3.0684182	3.0689190	3.1173572	3.2061842
Y0	4	2.9292390	3.0162474	3.0668149	3.0661296	3.1157539	3.2112047
Y0	agg	2.9274743	3.0183415	3.0676229	3.0664199	3.1160385	3.2089274

For each estimand category, there are five results (rows) that represent the output from each of the 4 MCMC chains and an aggregated output.

For visualization purposes, there are two options available as S3 methods for the `bartcs` object. The first option is the posterior inclusion probability (PIP) plot. PIP is the probability that a variable is used as a splitting variable, and can be interpreted as the importance of a variable. The `inclusion_plot()` function is a wrapper for the `bar_chart()` function from the `ggcharts` package, allowing the use of its arguments to customize the plot. The recommended arguments to use are `top_n` and `threshold`.

```
R> plot(separate_fit, method = "pip", top_n = 10)
```

```
R> plot(separate_fit, method = "pip", threshold = 0.5)
```

In Figure 4, the argument `top_n` allows you to select variables with the top `top_n` highest PIPs. The argument `threshold` displays variables with PIP greater than `threshold`. From a decision-theoretical perspective (Barbieri and Berger 2004; Linero 2018), variables with PIPs larger than 0.5 can be considered chosen confounders. It is worth noting that the five true confounders $X_1 - X_5$ are all correctly selected as true confounders with PIPs of 1, along with one extra predictor X_7 in the outcome model.

The second option for visualization is the traceplot, which is mainly used to check MCMC convergence. The function provides a traceplot of the average treatment effect (ATE) for each

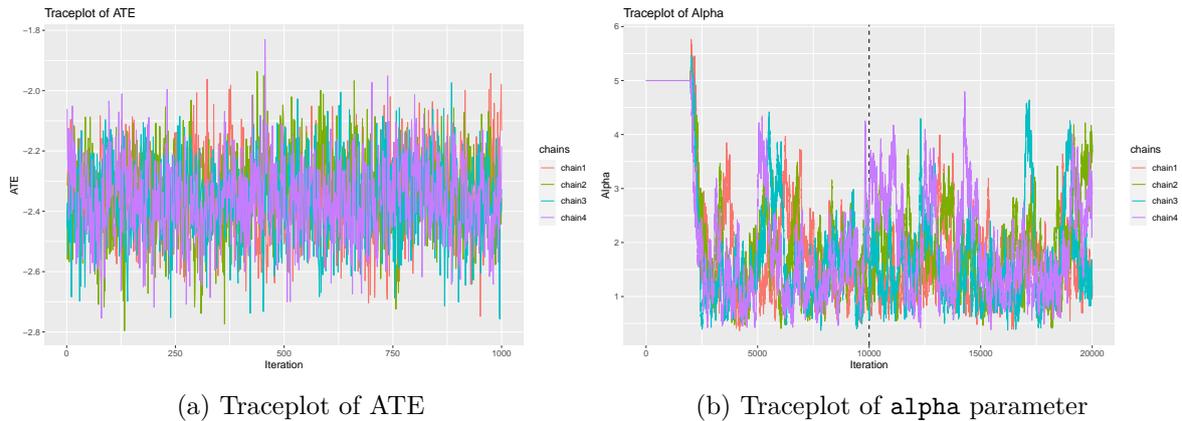


Figure 5: Traceplots for multiple MCMC chains

MCMC chain. Traceplots of other variables such as `dir_alpha` (the hyperparameter in the sparsity-inducing Dirichlet prior) and `sigma2_out` (the variance parameter in the outcome model) are also available by using the argument `parameter`.

```
R> plot(fit, method = 'trace')
```

```
R> plot(fit, method = 'trace', parameter = 'dir_alpha')
```

In Figure 5, the traceplots of the ATE and `dir_alpha` parameter are shown for four different MCMC chains. Visual inspection based on the traceplot is convenient, but for a more comprehensive assessment of convergence, it is recommended to use the gelman-rubin diagnostics provided in the `summary()`.

We evaluated the performance of **bartcs** in comparison to other models, including those generated by the **bacr** package that inspired our model development. The **bacr** package is easily installed via CRAN and loaded into the current R session as follows:

```
R> install.packages("bacr")
R> library("bacr")
```

To fit the model included in this package, we used the `bac()` function where the input data needs to be provided in the form of a data frame. To fit the exposure and outcome models in this case, a generalized linear model is used, and it is necessary to specify the family of the model based on the data type (e.g. `familyX="binomial"` and `familyY="gaussian"`). The MCMC algorithm was run for 10000 iterations after discarding the first 5000 iterations as burn-ins. Additionally, no interaction between the exposure and each confounder was assumed.

```
R> set.seed(42)
R> Z <- as.data.frame(cbind(Y, Trt, X))
R> fit.bac <- bac(
+   data = Z, exposure = "Trt", outcome = "Y",
+   confounders = paste("V", 3:(P+2), sep=""),
```

```
+   interactors = NULL, familyX = "binomial", familyY = "gaussian",
+   omega = Inf, num_its = 10000, burnM = 5000, burnB = 5000, thin = 10
+ )
```

The result can be checked through the `summary()` function as follows:

```
R> summary(fit.bac)
```

BAC objects:

Exposure effect estimate:

posterior mean	95% posterior interval
-1.6	(-2, -1.3)

Covariates with posterior inclusion probability > 0.5:

	posterior inclusion probability
V3	1.0000
V4	1.0000
V5	1.0000
V6	1.0000
V7	1.0000
V99	0.9248
V54	0.8398
V14	0.6964
V90	0.6575
V41	0.6514

The ATE posterior mean was estimated to be -1.6, which was significantly different from the true ATE value of -2.55. Moreover, the 95% credible interval (-2, -1.3) did not include the true value. When considering the importance of selected confounders based on the posterior inclusion probability, **bacr** included all important confounders $X_1 - X_5$ (that is, $V3 - V7$ in the summary), but also added $X_{12}, X_{39}, X_{52}, X_{88},$ and X_{97} (that is, $V14, V41, V54, V90, V99$ in the summary) with high PIPs, which were not true confounders. Notably, X_6 and X_7 , which are additional predictors of the outcome model, were not included. This result may be attributed to the fact that **bacr** relies on a parametric model and therefore may struggle to account for the non-linear and complex data structure.

5. Real Data Example

In the previous section, the `separate_bart()` function was used to demonstrate a separate outcome model scheme. In this section, a single outcome model is tested using the `single_bart()` function, based on the Infant Health and Development Program (IHDP) dataset as an example. This dataset was collected from a longitudinal study that tracked the development of low-birth-weight premature infants. The study participants in the treatment group received intensive care and home visits from trained providers and their cognitive test scores were evaluated at the end of the intervention period. The dataset includes a variety

Category	treatment=1 (n=139)		treatment=0 (n=608)	
	Mean	IQR	Mean	IQR
Y	6.43	(5.84, 7.34)	2.41	(1.45, 3.08)
X_1^*	0.21	(-0.39, 0.95)	-0.05	(-0.75, 0.79)
X_2^*	0.18	(-0.20, 0.59)	-0.04	(-0.60, 0.59)
X_3^*	-0.04	(-0.73, 0.38)	0.01	(-0.73, 0.76)
X_4^*	-0.22	(-0.88, 0.16)	0.05	(-0.88, 0.16)
X_5^*	-0.14	(-0.69, 0.56)	0.03	(-0.50, 0.68)
X_6^*	0.21	(-0.53, 0.96)	-0.05	(-0.86, 0.63)
X_7	0.52	(0, 1)	0.51	(0, 1)
X_8	0.09	(0, 0)	0.09	(0, 1)
X_9	0.68	(0, 1)	0.48	(0, 1)
X_{10}	0.29	(0, 1)	0.38	(0, 1)
X_{11}	0.25	(0, 1)	0.27	(0, 1)
X_{12}	0.22	(0, 0)	0.22	(0, 0)
X_{13}	0.38	(0, 1)	0.35	(0, 1)
X_{14}	1.58	(1, 2)	1.44	(1, 2)
X_{15}	0.14	(0, 0)	0.14	(0, 0)
X_{16}	0.94	(1, 1)	0.97	(1, 1)
X_{17}	0.69	(0, 1)	0.57	(0, 1)
X_{18}	0.99	(1, 1)	0.96	(1, 1)
X_{19}	0.15	(0, 0)	0.13	(0, 0)
X_{20}	0.06	(0, 0)	0.15	(0, 0)
X_{21}	0.17	(0, 0)	0.15	(0, 0)
X_{22}	0.04	(0, 0)	0.09	(0, 0)
X_{23}	0.01	(0, 0)	0.09	(0, 0)
X_{24}	0.06	(0, 0)	0.14	(0, 0)
X_{25}	0.27	(0, 1)	0.13	(0, 0)

Table 2: Summary statistics for the `ihdp` data set. \star denotes a continuous potential confounder.

of pretreatment variables, including 6 continuous and 19 binary covariates. Although the original IHDP dataset was used in causal research by Hill (2011), we will use a synthetic version of the IHDP dataset created by Louizos, Shalit, Mooij, Sontag, Zemel, and Welling (2017) which provides true values for comparison purposes. The data can be loaded by

```
R> set.seed(42)
R> data(ihdp, package = "bartcs")
```

and Table 2 displays the summary statistics of the variables. In the dataset, `y_factual` is the observed outcome Y (i.e., $Y(A)$) and `y_cfactual` is the counterfactual outcome Y (i.e., $Y(1 - A)$). We fit the single outcome model using the `single_bart()` function.

```
R> single_fit <- single_bart(
+   Y           = ihdp$y_factual,
+   trt        = ihdp$treatment,
```

```

+   X                = ihdp[, 6:30],
+   num_tree         = 10,
+   num_chain        = 4,
+   num_post_sample  = 100,
+   num_thin         = 10,
+   num_burn_in      = 1000
+ )
R> single_fit

```

```

`bartcs` fit by `single_bart()`

```

	mean	2.5%	97.5%
ATE	3.965814	3.746841	4.150017
Y1	6.386911	6.204699	6.551460
Y0	2.421097	2.336235	2.498595

The function `single_bart()` returns a `bartcs` object, which displays the posterior means and 95% credible intervals for the sample average treatment effect (ATE), and the potential outcomes $Y(1)$ and $Y(0)$. The `summary()` and `plot()` functions can also be used with this `bartcs` object generated by `single_bart()`.

```

R> summary(single_fit)

```

```

`bartcs` fit by `single_bart()`

```

Treatment Value

```

Treated group : 1
Control group : 0

```

Tree Parameters

Number of Tree	: 10	Value of alpha	: 0.95
Prob. of Grow	: 0.28	Value of beta	: 2
Prob. of Prune	: 0.28	Value of nu	: 3
Prob. of Change	: 0.44	Value of q	: 0.95

Chain Parameters

Number of Chains	: 4	Number of burn-in	: 1000
Number of Iter	: 2000	Number of thinning	: 10
Number of Sample	: 100		

Outcome Diagnostics

```

Gelman-Rubin : 0.9935952

```

Outcome

estimand	chain	2.5%	1Q	mean	median	3Q	97.5%
ATE	1	3.791494	3.939915	4.005987	4.018636	4.070101	4.215062
ATE	2	3.788716	3.908483	3.969369	3.987434	4.034971	4.120444

ATE	3	3.744850	3.899803	3.956803	3.961782	4.022265	4.137302
ATE	4	3.731310	3.865857	3.931098	3.928776	3.995207	4.145625
ATE	agg	3.746841	3.902287	3.965814	3.968083	4.037366	4.150017
Y1	1	6.235415	6.357685	6.422597	6.425478	6.482940	6.597327
Y1	2	6.238502	6.325691	6.393648	6.397596	6.457599	6.539438
Y1	3	6.209450	6.331860	6.378018	6.372846	6.441610	6.533386
Y1	4	6.187139	6.291756	6.353381	6.356149	6.408011	6.546361
Y1	agg	6.204699	6.325691	6.386911	6.383185	6.451711	6.551460
Y0	1	2.335569	2.386703	2.416610	2.413417	2.441407	2.494906
Y0	2	2.338024	2.388481	2.424279	2.427471	2.456748	2.508601
Y0	3	2.332171	2.393740	2.421215	2.424502	2.452225	2.487147
Y0	4	2.350610	2.400098	2.422283	2.424012	2.446279	2.492166
Y0	agg	2.336235	2.390741	2.421097	2.421593	2.449920	2.498595

We also fitted a separate outcome model to the `ihdp` data and compared the results with the single outcome model.

```
R> separate_fit <- separate_bart(
+   Y           = ihdp$y_factual,
+   trt         = ihdp$treatment,
+   X           = ihdp[, 6:30],
+   num_tree    = 10,
+   num_chain   = 4,
+   num_post_sample = 100,
+   num_thin    = 10,
+   num_burn_in = 1000
+ )
R> separate_fit
```

```
`bartcs` fit by `separate_bart()`

      mean      2.5%      97.5%
ATE 3.904472 3.669016 4.097428
Y1  6.329471 6.103259 6.507811
Y0  2.424999 2.343330 2.505997
```

As this is a simulated version of the IHDP data, the true values are known and are 4.02 for the average treatment effect (ATE), 6.45 for $E[Y(1)]$, and 2.43 for $E[Y(0)]$. The outputs from the two models accurately reflect these true values within their 95% credible intervals. Additionally, the PIP plots (Figure 6) show that both models selected the same three confounders. The important aspect here is that in the case of the single model, the exposure variable (`trt`) is also incorporated into the selection process. As indicated in Eq. (2), because the exposure variable is included as one of the covariates in the outcome model, it is subject to confounder selection. This means that in the computation of PIP, it is treated similarly to other confounders, producing the following plot (a) in Figure 6. In Figure 6, Plot (a) displays the potential confounders for the single outcome model, which have a posterior inclusion probability of 0.5 or more, while Plot (b) illustrates the confounders with a posterior inclusion

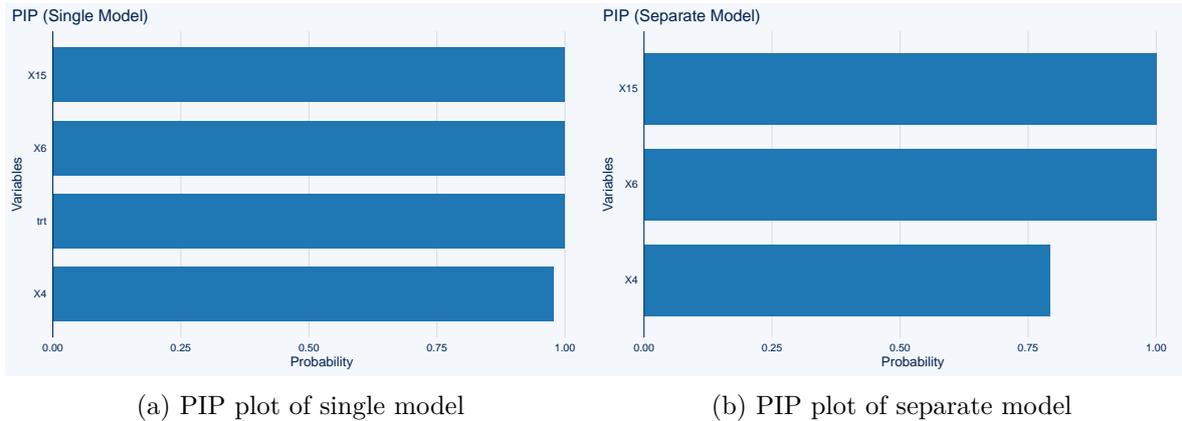


Figure 6: PIP plot for IHDP dataset

probability of 0.5 or more when the separate outcome model is used. It is noteworthy that X_4 , X_6 , and X_{15} were consistently chosen as confounders with PIP values larger than 0.5.

6. Continuous Exposure Example

When it comes to a continuous exposure variable, the formula in Eq. (1) is changed as follows:

$$A_i = \sum_{t=1}^T g_a(\mathbf{X}_i; \mathcal{T}_t, \mathcal{M}_t) + \epsilon_i, \epsilon_j \sim N(0, \tau^2).$$

This altered formula is used in conjunction with the single outcome model to perform confounder selection. However, the separate outcome model, which fits two distinct outcome models based on the two exposure levels, is not suitable for the continuous exposure variable. The **bartcs** has an advantage in handling continuous exposure through its `single_bart()` function. This function has the versatility to handle both binary and continuous treatments, and automatically identifies binary treatments when there are only two unique values. To demonstrate this, we generate a data set similar to the previous example.

```
R> set.seed(42)
R> N <- 300
R> P <- 100
R> cov <- list()
R> for(i in 1:P) {
+   cov[[i]] <- rnorm(N, 0, 1)
+ }
R> X <- do.call(cbind, cov)
R> h1 <- ifelse(X[, 1] < 0, 1, -1)
R> h2 <- ifelse(X[, 2] < 0, -1, 1)
R> mu_trt <- 0.5 + h1 + h2 - 0.5 * abs(X[, 3] - 1) + 0.5 * X[, 4] * X[, 5]
R> Trt <- rnorm(N, mu_trt, 0.3)
R> mu_y <- 1 * h1 + 1 * h2 - Trt + 1 * abs(X[, 3] + 1) +
+   1 * X[, 4] + exp(0.5 * X[, 5]) -
```

```
+ 0.5 * Trt * abs(X[, 6]) - 0.5 * Trt * abs(X[, 7] + 1)
R> Y <- rnorm(N, mu_y, 0.3)
R> treatment <- quantile(Trt, 0.75)
R> control <- quantile(Trt, 0.25)
```

We will use the function `single_bart()` to fit the generated data. The first and third quantile values of `Trt` will serve as the basis for comparing two different exposure levels. As arguments in `single_bart()`, we need to provide these two pre-specified exposure levels ($a = \text{trt_treated}$ and $a' = \text{trt_control}$). In this case, the causal estimand is $\Delta(a, a') = E[Y(a) - Y(a')]$.

```
R> single_fit <- single_bart(
+ Y = Y, trt = Trt, X = X,
+ trt_treated = treatment, trt_control = control,
+ num_tree = 50, num_chain = 4,
+ num_burn_in = 10000, num_thin = 10, num_post_sample = 1000
+ )
R> single_fit
```

```
`bartcs` fit by `single_bart()``
```

	mean	2.5%	97.5%
ATE	-2.799264	-4.1798117	-1.777905
Y1	1.075335	0.3445304	1.765200
Y0	3.874599	3.1925255	4.656736

Similar to other `bartcs` objects, the `summary()` and `plot()` functions can be applied to the continuous exposure scenario. Figure 7 displays a PIP plot, which demonstrates that out of 100 possible confounders, all of the true confounders except X_1 , X_2 , and two additional predictors were captured effectively, with high PIP values.

7. Summary and discussion

In conclusion, the `bartcs` R package is a powerful tool for causal inference using BART. It allows users to adjust for confounders and estimate treatment effects using a flexible non-parametric method. The package's ability to handle high-dimensional and non-linear confounding, binary treatments, and continuous treatments makes it a versatile tool for a wide range of applications. Additionally, the package's support for parallel computing and visualization of results make it a user-friendly and easy-to-interpret tool. The `bartcs` package is a valuable resource for researchers in various fields.

Computational details

The results in this paper were obtained using R 4.2.2 on a MacBook Air with a M1 chip and 16 GB of memory. `bartcs` 1.0.0 and `bacr` 1.0.1 were used for the analysis. R itself

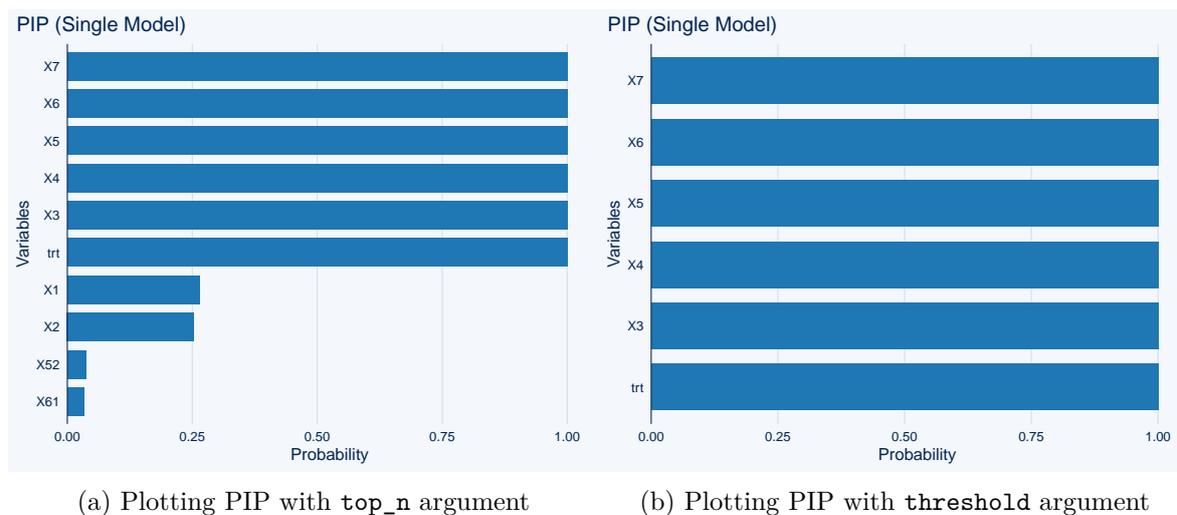


Figure 7: PIP plot for continuous exposure

and all packages used are available from the Comprehensive R Archive Network (CRAN) at <https://CRAN.R-project.org/>.

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A. Posterior Computation

We use ‘‘Bayesian backfitting’’ (Hastie *et al.* 2000) to obtain posterior samples from

$$P(\mathcal{T}'_1, \dots, \mathcal{T}'_T, \mathcal{M}'_1, \dots, \mathcal{M}'_T, \sigma^2 | \mathbf{D})$$

for the outcome model (2) (or (3)). This involves a Metropolis-within-Gibbs sampler, where we fit each tree \mathcal{T}'_t iteratively using residual responses:

$$R_{i,-t} = y_i - \sum_{j \neq t} g_y(\mathbf{X}_i; \mathcal{T}'_j, \mathcal{M}'_j)$$

for $i = 1, \dots, N$. For each tree t , we propose a new tree structure \mathcal{T}'_t from the full conditional $[\mathcal{T}'_t | R_{1,-t}, \dots, R_{n,-t}, \sigma^2]$ (i.e., grow, prune or change alterations), and update the parameter within the tree through the full conditional $[\mathcal{M}'_t | \mathcal{T}'_t, R_{1,-t}, \dots, R_{n,-t}, \sigma^2]$.

To draw samples from $P(\mathcal{M}'_t | \mathcal{T}'_t)$, we assume a prior $\mu \sim N(\mu_\mu/T, \sigma_\mu^2)$ on each of the leaf parameters $\mathcal{M}'_t = \{\mu_1, \mu_2, \dots, \mu_{t_b}\}$, where t_b is the number of terminal nodes in tree \mathcal{T}'_t . The range center of the outcome, μ_μ , is set as the mean of the range of the outcome, and σ_μ^2 is empirically determined to satisfy $T\mu_\mu - 2\sqrt{T}\sigma_\mu = y_{\min}$ and $T\mu_\mu + 2\sqrt{T}\sigma_\mu = y_{\max}$ (Kapelner and Bleich 2016). indices corresponding to the η th terminal node and the number of observations in that node, respectively.

We generate a sample μ_η from the posterior distribution for the η -th terminal node in tree \mathcal{T}'_t by using the following equation:

$$\mu_\eta \sim N \left(\frac{1}{1/\sigma_\mu^2 + n_\eta/\sigma^2} \left(\frac{\mu_\mu/T}{\sigma_\mu^2} + \frac{\sum_{i \in I_\eta} R_{i,-t}}{\sigma^2} \right), \left(\frac{1}{\sigma_\mu^2} + \frac{n_\eta}{\sigma^2} \right)^{-1} \right),$$

where I_η and n_η correspond to the observation indices and the number of observations, respectively, for the η -th terminal node.

To use the separate model scheme, we perform a backfitting step to draw samples from $P(\mathcal{T}'_1, \dots, \mathcal{T}'_T, \mathcal{M}'_1, \dots, \mathcal{M}'_T, \sigma_a^2 | \mathbf{D})$ for each $A = a \in \{0, 1\}$ by computing the residual responses iteratively as follows:

$$R_{i,-t} = y_i - \sum_{j \neq t} g_y^a(\mathbf{X}_i; \mathcal{T}'_j, \mathcal{M}'_j) \quad \text{for } i \in I_a,$$

where I_a represents the set of observations corresponding to $A = a \in \{0, 1\}$.

To obtain posterior samples from $P(\mathcal{T}_1, \dots, \mathcal{T}_T, \mathcal{M}_1, \dots, \mathcal{M}_T | \mathbf{D})$ for the binary exposure model (1), we introduce a latent variable Z and apply the general Bayesian additive regression tree (BART) model for continuous data. Specifically, we define Z_i for $i = 1, \dots, n$ as:

$$Z_i \sim \begin{cases} N \left(\sum_{t=1}^T g_a(\mathbf{X}_i; \mathcal{T}_t, \mathcal{M}_t), 1 \right) I_{(Z_i > 0)} & \text{for } A_i = 1; \\ N \left(\sum_{t=1}^T g_a(\mathbf{X}_i; \mathcal{T}_t, \mathcal{M}_t), 1 \right) I_{(Z_i \leq 0)} & \text{for } A_i = 0 \end{cases}$$

where $g_a(\mathbf{X}_i; \mathcal{T}_t, \mathcal{M}_t)$ denotes the function that maps \mathbf{X}_i to a predicted value based on the t -th tree and the t -th model associated with exposure level a . For continuous A , the updating step follows the Bayesian backfitting method as described for model (2).

Once all the tree structures and corresponding parameters have been updated, we proceed to update the variance parameters (σ^2 in the outcome model (2)) using the Gibbs sampler and the final residuals. This is achieved by sampling from the inverse gamma distribution given by:

$$\sigma^2 \sim \text{Inv.Gamma} \left(a_\sigma + \frac{N}{2}, b_\sigma + \frac{1}{2} \left\{ \sum_{i=1}^N \left(y_i - \sum_{t=1}^T g_y(\mathbf{X}_i; \mathcal{T}_t', \mathcal{M}_t') \right) \right\} \right),$$

where $a_\sigma = b_\sigma = 3$ is set as suggested in [Chipman *et al.* \(2010\)](#). In the case of the separate model, two variance parameters (σ_0^2, σ_1^2) need to be updated for the two outcome models (i.e., Model (3) for $A = a \in \{0, 1\}$), which is done using the inverse gamma distribution as follows:

$$\sigma_a^2 \sim \text{Inv.Gamma} \left(a_\sigma + \frac{N_a}{2}, b_\sigma + \frac{1}{2} \left\{ \sum_{i \in I_a} \left(y_i - \sum_{t=1}^T g_y^a(\mathbf{X}_i; \mathcal{T}_t^a, \mathcal{M}_t^a) \right) \right\} \right),$$

where N_a is the number of observations under $A = a$. If A is continuous, the variance parameter for the exposure model, τ^2 is updated in a similar manner:

$$\tau^2 \sim \text{Inv.Gamma} \left(a_\tau + \frac{N}{2}, b_\tau + \frac{1}{2} \left\{ \sum_{i=1}^N \left(A_i - \sum_{t=1}^T g_a(\mathbf{X}_i; \mathcal{T}_t^*, \mathcal{M}_t^*) \right) \right\} \right),$$

where $a_\tau = b_\tau = 3$.

Next, we update the parameter α in the prior distribution of selection probabilities $\mathbf{s} \sim \mathcal{D}(\alpha/q, \dots, \alpha/q)$ by selecting a prior of the form $\alpha/(\alpha + q) \sim \text{Beta}(a_0, b_0)$, where $a_0 = 0.5$ and $b_0 = 1$, as suggested in [Linero \(2018\)](#). The Metropolis-Hastings algorithm is then used to update the parameter. For the single model scheme, we update the vector of selection probabilities \mathbf{s} using the Metropolis-Hastings algorithm, with the acceptance ratio given by

$$P(\mathbf{s} \rightarrow \mathbf{s}^{\text{new}}) = \min \left\{ 1, \left[\frac{(1 - \sum_{j=1}^q s_j) / (1 - \sum_{j=1}^q s_j^{\text{new}})}{\sum_{j=1}^J n_j^a} \right]^{\sum_{j=1}^J n_j^a} \right\}.$$

In this step, the proposal distribution for \mathbf{s} is given as $\mathcal{D}(n_0^y + c + \alpha/q, n_1^a + n_1^y + \alpha/q, n_2^a + n_2^y + \alpha/q, \dots, n_q^a + n_q^y + \alpha/q)$. For the separate model approach, we update \mathbf{s} using a conjugate sampling update as follows: $\mathbf{s} \sim \mathcal{D}(\alpha/q + n_1^a + n_1^{y1} + n_1^{y0}, \dots, \alpha/q + n_q^a + n_q^{y1} + n_q^{y0})$, where n_j^{y1} and n_j^{y0} represent the number of splits on the confounder X_j in two separate outcome models.

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