Introduction to lifecontingencies Package

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Abstract

lifecontingencies performs actuarial present value calculation for life insurances. This paper briefly recapitulate the theory regarding life contingencies (life tables, financial mathematics and related probabilities) on life contingencies. Then it shows how **lifecontingencies** functions represent a perfect cookbook to perform life insurance actuarial analysis and related stochastic simulations.

Keywords: life tables, financial mathematics, actuarial mathematics, life insurance, R.

1. Introduction

As of March 2012, **lifecontingencies** appears to be the first R package that deals with life insurance evaluation. Some actuarial packages have been already available in R, however most of these packages mainly interest non-life actuaries. In fact non - life insurance modeling uses more data analysis and applied statistical modelling than life insurance does. E.g. functions to fit loss distributions and to perform credibility analysis are provide within the package **actuar**, Dutang, Goulet, and Pigeon (2008). Package **actuar** represents the computational side of the classical actuarial manual Loss Distribution, Klugman, Panjer, Willmot, and Venter (2009). The package **ChainLadder**, Gesmann and Zhang (2011), provides functions to estimate non-life unpaid loss reserve. GLM models, widely used in non - life insurance pricing, can be fit by functions bundled in the base R distribution. More advanced predictive models used by actuaries, e.g. GAMLSS and Tweedie regressions, can be fit using specifically developed packages as **gamlss**, Rigby and Stasinopoulos (2005), and **cplm** packages respectively. Life insurance evaluation models demographic and financial data mainly . R has a dedicated view to packages specifically tailored to financial analysis. But, few packages that handle demographic data have been published yet. Relevant packages that can aid demographers work are **demography**, Rob J Hyndman, Heather Booth, Leonie Tickle, and John Maindonald (2011), and **LifeTables**, Riffe (2011). Packages **YieldCurve**, Guirreri (2010), and **termstrc**, Ferstl and Hayden (2010), can be used to perform interest rate analysis. Finally no package exists that performs life contingencies calculations, as of February 2012.

Numerous commercial software specifically tailored to actuarial analysis are available in commerce. Moses and Prophet are currently the leading actuarial software for life insurance modelling. **lifecontingencies** package aims to represent the R computational side of the concepts exposed in the classical Society of Actuaries Actuarial Mathematics book, Bowers, Gerber, Hickman, Jones, and Nesbitt (1997). Since life contingencies theory grounds on demography and classical financial mathematics, I have made use of the Ruckman and Francis Ruckman and Francis (2006) and Broverman Broverman (2008) as references. The structure of the vignette document is:

1. Section 2 outlines the statistical and financial mathematics theory regarding life contingencies.
2. Section 3 overviews the structure of the **lifecontingencies** package.
3. Section 4 gives a wide choice of applied **lifecontingencies** examples .
4. Finally Section 5 discusses package actual and prospective development and known limitations.

2. Life contingencies statistical and financial foundations

Life insurance analysis involves the calculation of statistics regarding occurrence and amount of future cash flows. E.g. the insurance pure premiun (also known as benefit premium) is the present value of the series of future cash flows whose probability is based on the occurrence of the policyholder's life events (life contingencies). Therefore life insurance actuarial mathematics grounds itself on concepts derived from demography and the theory of interest.

A life table (also called a mortality table or actuarial table) is a table that shows how mortality affects subject of a cohort across different ages. It reports for each age x, the number of subjects l_x living at the beginning of age x. It represents a sequence of $l_0, l_1, \ldots, l_\omega$, where ω is the farthest age until which a subject of the cohort can survive. Life table are tipically distinguished according to gender, year of birth and nationality.

Using a statistical perspective, a life table allows the probability distribution of the the future lifetime for a subject aged x, to be deduced. In particular a life table allows to derive two key probability distribution: T_x, the future lifetime for a subject aged x and its curtate form, K_x, i.e., the number of future years completed before death. Therefore many statistics can be derived from the life table. A non exhaustive list follows:

- ${}_tp_x = \frac{l_{x+t}}{l_x}$, the probability that someone living at age x will reach age $x+t$.
- ${}_tq_x$, the complementary probability of ${}_tp_x$.
- ${}_td_x$, the number of deaths between age x and $x+t$.
- ${}_tL_x = \sum_{t=0}^{n} l_{x+t}$, the expected number of years lived by the cohort between ages x and $x+t$.
- ${}_tm_x = \frac{{}_td_x}{{}_tL_x}$, the central mortality rate between ages x and $x+t$.
- e_x, the curtate expectation of life for a subject aged x, $e_x = E\left(K_x\right)$ and its complete form $\underset{x}{\overset{\circ}{e}} = E\left(T_x\right)$.

The Keyfitz manual, Keyfitz and Caswell (2005), provides an exaustive coverage about life table theory and practice. Life table are usually published by institutions that have access to large amount of reliable historical data, like government statistics or social security bureaus. It is a common practice for actuaries to start from these life tables and to adapt them to the insurer's portfolio actual experience.

Classical financial mathematics deals with monetary amount that could be available in different times. The present value of a series of cash flows, reported in formula 2, is probably the most important concept. The present value represents the current value of a series of monetary cash flows, CF_t, that will be available in different periods of time.
The interest rates, i_t, represents the measure of the price of money available in future times. This paper will use i to express the effective (real) compound interest. It means that if i is the interest rate, a sum of 1 monetary unit accumulates throught time according to the law $A(t) = (1+i)^t$, being $A(t)$ the accumulation function. Arrangements lead to discount and nominal (m-compound) interest rates as shown by equation 1.

$$A(t) = (1+i)^t = (1-d)^{-t} = \left(1+\frac{i^m}{m}\right)^{t*m} = \left(1-\frac{d^m}{m}\right)^{-t*m} \tag{1}$$

All financial mathematics functions (as annuities, $\bar{a}_{\overline{n}|}$, or accumulated values, $s_{\overline{n}|}$) can be written as a particular case of formula 2. See the classical Broverman (2008) manual for further reference on the topic.

$$PV = \sum_{t \in T} CF_t (1+i_t)^{-t} \tag{2}$$

Actuaries uses the probabilities inherent the life table to evaluate life contingencies insurances. Life contingencies are themselves stochastic variables, in fact. They consist in present values whose amounts are not certain, since the time of their occurrence and the final values depend by events regarding the life of the insured head. **lifecontingencies** package provides function to model many of such random variables, $\tilde{Z}$, and in particular their expected value, the Actuarial Present Value (APV). APV is certainly the most important statistic of $\tilde{Z}$ variables that actuaries use. It represents the average cost of the benefit the insurer provides to the policyholder. Insured benefits and loadings for expense and profit adds to the final premium proposed to policyholders. Life contingencies can be either continue or discrete. **lifecontingencies** package models only discrete life contingencies, that is insured amounts are supposed to be due at the end of each year or fraction of year. The **?** manual contains formulas to obtain continue life contingencies APV from the corresponding discrete forms.

Few examples of life contingencies follow:

1. An n-year term life insurance provides payment of b if the insured dies within n years from issue. If the payment is perfomed at the end of year of death, we can write $\tilde{Z}$ as $\tilde{Z} = \begin{cases} b * v^{\tilde{K}_x+1}, \tilde{K}_x \leq n \\ 0, \tilde{K}_x > n \end{cases}$ The APV symbol is $A^1_{x:\overline{n}|}$.

2. A life annuity consists in a series of benefits paid contingent upon survival of a given life. In particular, a temporary life annuity due pays a benefit at the beginning of each

period so long as the annuitant (x) survives, for up to a total of n years, or n payments. Assuming \$1 payments, we can write $\tilde{Z}$ as $\tilde{Z} = a_{\overline{\tilde{K}+1}|}$. Its APV expression is $\ddot{a}_{x:\overline{n}|}$.

3. An n-year pure endowment insurance grants a benefit payable at the end of n years if the insured survives at least n years from issue. The expression of $\tilde{Z}$ is $v^n * I\left(\tilde{K}_x \geq n\right)$ and its APV expression is ${}_nE_x$.

We remaind to the Bowers *et al.* (1997) manual for formulas regarding other life contingencies insurances as $(DA)^1_{x:\overline{n}|}$, the decreasing term life insurance, $(IA)_x$, the increasing term life insurance, and common variations on benefit payments: deferrment and fractional amounts.

The **lifecontingencies** package provides functions that allows the actuary to evaluate the APV and to draw random samples from $\tilde{Z}$ distribution. Three approach have been traditionally followed for the evaluation of the APV: the use of commutation tables, the current payment technique and the expected value tecniques. Commutation tables extend life table by tabulating special function of age and rate of interest whose ratios allow the actuary to evaluate APV for standard insurances, as discussed in Anderson (1999). The **lifecontingencies** allows underlying commutation table to be printed out as further described. However commutation table usage has become useless in computer era since it is not enough flexible and it is computationally inefficient. Therefore commutation table approach has not been used within **lifecontingencies** to perform APV calculations.

The current payment technique calculates the APV, $\bar{z}$, as the scalar product of three vectors: $\bar{z} = \langle\langle\bar{c} \bullet \bar{v}\rangle \bullet \bar{p}\rangle$. The vector of uncertain cash flows, $\bar{c}$, the vector of discount factors, $\bar{v}$ and the vector of cash flow probability. Since the current payment technique is the the most efficient approach from a computationally side perspective, we have used this approach to evaluate APV. Finally, the expected value approach models the APV as the scalar product of two vector: $\Pr\left[\tilde{K} = k\right]$, the probability that the future curtate lifetime is exactly k, where $\tilde{K}_x = 0, \ldots, \omega - x$, the present value of the benefit due under the insurance terms if the future curtate lifetime is exaclty k. The latter approach has been used to define the probability distribution of $\tilde{Z}$ in order to allow random sampling from its distribution.

3. The structure of the package

Package **lifecontingencies** contains classes and methods to handle lifetables and actuarial tables conveniently.

The package is loaded within the R command line as follows:

```
R> library(lifecontingencies)
```

Two main S4 classes Chambers (2008) have been defined within the **lifecontingencies** package: the `lifetable` class and the `actuarialtable` class. The lifetable class is defined as follows

```
R> #definition of lifetable
R> showClass("lifetable")

Class "lifetable" [in ".GlobalEnv"]

Slots:

Name:          x        lx      name
Class:   numeric   numeric character

Known Subclasses: "actuarialtable"
```

Class `actuarialtable` inherits from `lifetable` class. It has an additional slots for the interest rate.

```
R> showClass("actuarialtable")

Class "actuarialtable" [in ".GlobalEnv"]

Slots:

Name:   interest         x        lx      name
Class:   numeric   numeric   numeric character

Extends: "lifetable"
```

Beyond generic S4 classes and method there are three groups of functions: demographics functions, financial mathematics functions and actuarial mathematics functions.

The demographic functions group comprises the followings:

1. `dxt` returns deaths between age x and $x+t$, $d_{x,t}$.

2. `pxt` returns survival probability between age x and $x+t$, $p_{x,t}$.

3. `pxyt` returns the survival probability for two lifes, $d_{xy,t}$.
4. `qxt` returns death probability between age x and $x+t$, $q_{x,t}$.
5. `qxyt` returns the survival probability for two lifes, $q_{xy,t}$.
6. `Txt` returns the number of person-years lived after exact age x, $T_{x,t}$.
7. `mxt` returns central mortality rate, $m_{x,t}$.
8. `exn` returns the complete or curtate expectation of life from age x to $x+n$, $e_{x,n}$.
9. `rLife` returns a sample from the time until death distribution underlying a life table.
10. `exyt` returns the expected life time for two lifes between age x and $x+t$.
11. `probs2lifetable` returns a life table l_x from raw one - year survival / death probabilities.

The financial mathematics group comprises the followings:

1. `presentValue` returns the present value for a series of cash flows, $PV = \sum_i CF_i * v^{t_i}$.
2. `annuity` returns the present value of a annuity - certain, $a_{\overline{n}|}$.
3. `increasingAnnuity` returns the present value of an increasing annuity - certain, $(IA)_n$.
4. `accumulatedValue` returns the future value of a series of cash flows, $s_{\overline{n}|}$.
5. `decreasingAnnuity` returns the present value of an increasing annuity, $(DA)_{\overline{n}|}$.
6. `accumulatedValue` returns the future value of a payments sequence, $s_{\overline{n}|}$.
7. `nominal2Real` returns the effective annual interest (discount) rate i given the nominal m-periodal interest $i^(k)$ or discount $d^(k)$ rate.
8. `real2Nominal` returns the m-periodal interest or discount rate given the m periods or the discount.
9. `intensity2Interest` returns the intensity of interest δ given the interest rate i.
10. `interest2Intensity` returns the interest rate i given the intensity of interest δ.
11. `duration` returns the duration of a series of cash flows, $\sum_t^T \frac{t*CF_t\left(1+\frac{i}{m}\right)^{-t*m}}{P}$.
12. `convexity` returns the convexity of a series of cash flows, $\sum_t^T t*\left(t+\frac{1}{m}\right)*CF_t\left(1+\frac{y}{m}\right)^{-m*t-2}$.

The actuarial mathematics group comprises the following functions, for which we report must important function:

1. `Axn` models one head life insurance, whose APV symbol is $A^1_{x:\overline{n}|}$.

2. `Axyn` models two heads life insurances, whose APV symbol is $\bar{A}_{\overset{1}{xy}:\overline{n}|}$.
3. `axn` models annuities, whose APV symbol is $\ddot{a}_x$.
4. `axyn` models two heads annuities, whose APV symbol is $\ddot{a}_{xy}$.
5. `Exn` models pure endowment, whose APV symbol is ${}_nE_x$.
6. `Iaxn` models the increasing annuity, whose APV symbol is $(Ia)_x$.
7. `IAxn` models the increasing life insurance, whose APV symbol is $(IA)_x$.
8. `DAxn` models the decreasing life insurance, whose APV symbol is $(DA)_x$.

As general remark, standard financial and actuarial mathematics functions parameters are:

- `x`, the policyholder's age at the policy issuance time.
- `n`, the coverage duration that could be missing if the policy lasts for the remaining lifetime. For financial mathematics function it represent the lenght of the payment.
- `actuarialtable`, an actuarial table on which life insurance calculation are performed.
- `i`, the interest rate that overrides the `actuarialtable` default interest rate.
- `k`, the frequency of payments per year (default value is 1).

4. Code and examples

4.1. Classical financial mathematics example

The **lifecontingencies** package provides functions for classical financial mathematics calculation.
Following examples will show how to handle interest and discount rates with different compounding frequency, how to perform present value, annuities and future values analysis calculations as long as loans amortization and bond pricing.

Interest rate functions

Following examples show how to switch from effective interest rates (APR) to nominal interest rates, that is $i^k \to i$ and vice versa.

```
R> #an APR of 3% is equal to a
R> real2Nominal(0.03,12)

[1] 0.02959524

R> #of nominal interest rate while
R> #6% annual nominal interest rate is the same of
R> nominal2Real(0.06,12)

[1] 0.06167781

R> #while
R>
R> #4% effective interest rate corresponds to
R> real2Nominal(0.04,4)*100

[1] 3.941363

R> #nominal interest rate (in 100s) compounded quarterly
```

`real2Nominal` and `nominal2Real` work with discount rates also, that is they allows $d^(m) \to d$ and vice versa.

```
R> #an effective rate of discount of 4% is equal to a
R> real2Nominal(i=0.04,k=12,type="discount")

[1] 0.04075264

R> #nominal rate of discount payable quarterly
```

Present value analysis

Performing a project appraisal means evaluating the net present value (NPV) of all projected cash flows, as code below shows:

```
R> #suppose an investment requires an initial outflows of 1000
R> #and suppose its projected inflows to be 200, 500 and 700
R> capitals=c(-1000,200,500,700)
R> #at times.
R> times=c(0,1,2,5)
R> #the net prepreset value of the investment is
R> presentValue(cashFlows=capitals, timeIds=times,
+              interestRates=0.03)

[1] 269.2989

R> #@ 3% interest rate
R>
R>  #while if interest rates were time - varying
R> #e.g. 4% 2% 3% 5.7%
R> iRates=c( 0.04, 0.02, 0.03, 0.057)
R> presentValue(cashFlows=capitals, timeIds=times,
+              interestRates=iRates)

[1] 197.9225

R> #and if the last cash flow were uncertain, as we assume a
R> #probability to receive it of 50%, all other being equal
R> #the expected APV is
R> presentValue(cashFlows=capitals, timeIds=times,
+  interestRates=iRates,
+              probabilities=c(1,1,1,0.5))

[1] -67.35058
```

Annuities and future values

Code below shows examples of annuities ($a_{\overline{n}|}$) and accumulated values ($s_{\overline{n}|}$) evaluations.

```
R> #the PV of an annuity immediate $100 payable at the end of next 5 years at 3%
R> 100*annuity(i=0.03,n=5)

[1] 457.9707

R> #while the corresponding future value is
R> 100*accumulatedValue(i=0.03,n=5)
```

```
[1] 530.9136
```

While a more real-life example is shown below

```
R> #A man wants to save 100,000 to pay for the education
R> #of his son in 10 years time. An education fund requires the investors to
R> #deposit equal instalments annually at the end of each year. If interest of
R> #0.05 is paid, how much does the man need to save each year (R) in order to
R> #meet his target?
R> C=100000
R> R=C/accumulatedValue(i=0.05,n=10)
R> R

[1] 7950.457
```

while the code below shows how fractional annuities ($a^{(m)}_{lcroofn}$) can be handled when using `annuity` and `accumulatedValue` functions.

```
R> #Find the present value of an annuity-immediate of
R> #100 per quarter for 4 years, if interest is compounded semiannually at
R> #the nominal rate of 6%.
R> #the APR is
R> APR=nominal2Real(0.06,2)
R> 100*4*annuity(i=APR,n=4,k=4)

[1] 1414.39
```

Finally `increasingAnnuity` and `decreasingAnnuity` functions handle increasing ($(IA)_x$) and decreasing ($(DA)_x$) annuities.

```
R> #An increasing n-payment annuity-due shows payments of 1, 2, ... , n
R> #at time 0, 1, ... ,
R> #n - 1 . At interest rate of
R> #0.03 and n=10, its present value of the annuity is
R> increasingAnnuity(i=0.03, n=10,type="due")

[1] 46.18416

R> #while the present value of a decreasing
R> #annuity due of 10, 9,...,1
R> #from time 1 to time 10 is
R> decreasingAnnuity(i=0.03, n=10,type="immediate")

[1] 48.99324
```

Finally the calculation of the present value of a geometrically increasing annuity is shown in the code below

```
R> #assuming each year the annuity increases its value by 3%
R> #being the interest rate is 4%
R> #and the annuity duration being 10 years
R>
R> #first determine the effective interest rate
R> ieff=(1+0.04)/(1+0.03)-1
R> annuity(i=ieff,n=10)

[1] 9.48612
```

Loan amortization

The code below shows an investment amortization payment schedule. Suppose loaned capital is C, then assuming an interest rate i, the amount due to the lender at each instalment is $R = \frac{C}{a_{\overline{n|}}}$. At each installment the R_t amount repays $I_t = C_{t-1} * i$ as interest and $C_t = R_t - I_t$ as capital.

```
R> #assumeing: 5% annual interest rate
R> #a capital of 100,000
R> #two payments per year
R>
R> capital=100000
R> interest=0.05
R> payments_per_year=2 #payments per year
R> rate_per_period=(1+interest)^(1/payments_per_year)-1
R> years=5 #five years length of the loan
R> installment=1/payments_per_year*capital/annuity(i=interest, n=years,k=payments_per_year
R> installment

[1] 11407.88

R> #compute the balance due at the beginning of each period
R> balance_due=numeric(years*payments_per_year)
R> balance_due[1]=capital*(1+rate_per_period)-installment
R> for(i in 2:length(balance_due))
+  {
+          balance_due[i]=balance_due[i-1]*(1+rate_per_period)-installment
+          cat("Payment ",i, " balance due:",round(balance_due[i]),"\n")
+  }

Payment  2  balance due: 81903
Payment  3  balance due: 72517
Payment  4  balance due: 62900
Payment  5  balance due: 53046
Payment  6  balance due: 42948
Payment  7  balance due: 32600
```

```
Payment  8  balance due: 21998
Payment  9  balance due: 11133
Payment  10  balance due: 0
```

Bond pricing

Bond pricing is another application of present value analysis. A standard bond whose principal will be repaid at time T is a series of coupon c_t, priced according to a coupon rate $j^(k)$ on a principal C. Formula 3 expresses the present value of a bond.

$$B_t = c_t a^{(k)}{}_{\overline{n|}} + Cv^T \tag{3}$$

We will show how to evaluate a standard bond with following examples:

```
R> #define a function to compute bond market value
R> #using functions provided within lifecontingencies package
R> bond<-function(faceValue, couponRate, couponsPerYear, yield,maturity)
+  {
+          out=NULL
+          numberOfCF=maturity*couponsPerYear #determine the number of CF
+          CFs=numeric(numberOfCF)
+          payments=couponRate*faceValue/couponsPerYear #determine the coupon sum
+          cf=payments*rep(1,numberOfCF)
+          cf[numberOfCF]=faceValue+payments #set the last payment amount
+          times=seq.int(from=1/couponsPerYear, to=maturity, by=maturity/numberOfCF)
+          out=presentValue(cashFlows=cf, interestRates=yield, timeIds=times)
+          return(out)
+  }
R> #coupon rate 6%, two coupons per year, face value 1000,
R> #yield 5%, three years to maturity
R> bond(1000,0.06,2,0.05,3)

[1] 1029.25

R> #coupon rate 3%, one coupons per year,
R> #face value 1000, yield 3%, three years to maturity
R> bond(1000,0.06,1,0.06,3)

[1] 1000
```

4.2. Lifetables and actuarial tables analysis

`lifetable` and `actuarialtable` classes are designed to handle demographics and actuarial mathematics calculations. A `actuarialtable` class inherits from `lifetable` class. It has one more slot that sets the rate of interest. Both classes have been designed using the S4 R classes framework.
Following examples show how these classes are initialized, basic survival probabilities and life tables analysis.

Creating lifetable and actuarialtable objects

Lifetable objects can be created by raw R commands or using existing `data.frame` objects. However, to build a `lifetable` class object three components are needed:

1. The years sequence, that is an integer sequence $0, 1, \ldots, \omega$. It shall starts from zero and going to the ω age (the age x that $p_x = 0$).
2. The l_x vector, that is the number of subjects living at the beginning of age x.
3. The name of the life table.

To create a `lifetable` object directly we can do as follows

```
R> x_example=seq(from=0,to=9, by=1)
R> lx_example=c(1000,950,850,700,680,600,550,400,200,50)
R> exampleLt=new("lifetable",x=x_example, lx=lx_example, name="example lifetable")
```

`print` and `show` methods are available that tabulate the x, lx, px and ex values.

```
R> print(exampleLt)

Life table example lifetable

  x   lx        px       ex
1 0 1000 0.9500000 4.742105
2 1  950 0.8947368 4.241176
3 2  850 0.8235294 4.042857
4 3  700 0.9714286 3.147059
5 4  680 0.8823529 2.500000
6 5  600 0.9166667 1.681818
7 6  550 0.7272727 1.125000
8 7  400 0.5000000 0.750000
9 8  200 0.2500000 0.500000
```

`head` and `tail` methods for `data.frame` S3 classes have also been implemented on `lifetable` classes, as shown below.

```
R> #head method
R> head(exampleLt)
```

```
  x   lx
1 0 1000
2 1  950
3 2  850
4 3  700
5 4  680
6 5  600

R> #tail method
R> tail(exampleLt)

   x  lx
5  4 680
6  5 600
7  6 550
8  7 400
9  8 200
10 9  50
```

Nevertheless the easiest way to create a `lifetable` object is to start from a suitable existing `data.frame`.

```
R> #load USA Social Security LT
R> data(demoUsa)
R> usaMale07=demoUsa[,c("age", "USSS2007M")]
R> usaMale00=demoUsa[,c("age", "USSS2000M")]
R> #coerce from data.frame to lifecontingencies
R> #requires x and lx names
R> names(usaMale07)=c("x","lx")
R> names(usaMale00)=c("x","lx")
R> #apply coerce methods and changes names
R> usaMale07Lt<-as(usaMale07,"lifetable")
R> usaMale07Lt@name="USA MALES 2007"
R> usaMale00Lt<-as(usaMale00,"lifetable")
R> usaMale00Lt@name="USA MALES 2000"
R> #create the tables
R> ##males
R> lxIPS55M<-with(demoIta, IPS55M)
R> pos2Remove<-which(lxIPS55M %in% c(0,NA))
R> lxIPS55M<-lxIPS55M[-pos2Remove]
R> xIPS55M<-seq(0,length(lxIPS55M)-1,1)
R> ##females
R> lxIPS55F<-with(demoIta, IPS55F)
R> pos2Remove<-which(lxIPS55F %in% c(0,NA))
R> lxIPS55F<-lxIPS55F[-pos2Remove]
R> xIPS55F<-seq(0,length(lxIPS55F)-1,1)
R> #finalize the tables
```

```
R> ips55M=new("lifetable",x=xIPS55M, lx=lxIPS55M,
+                   name="IPS 55 Males")
R> ips55F=new("lifetable",x=xIPS55F, lx=lxIPS55F,
+                   name="IPS 55 Females")
```

The last way a lifetable object can be created is to generate it from one year survival or death probabilities. This feature is useful when used in conjunction with the results of a mortality projection method (e.g. Lee - Carter).

```
R> #use 2002 Italian males life tables
R> data(demoIta)
R> itaM2002<-demoIta[,c("X","SIM92")]
R> names(itaM2002)=c("x","lx")
R> itaM2002Lt<-as(itaM2002,"lifetable")

removing NA and 0s

R> itaM2002Lt@name="IT 2002 Males"
R> #reconvert in data frame
R> itaM2002<-as(itaM2002Lt,"data.frame")
R> #add qx
R> itaM2002$qx<-1-itaM2002$px
R> #reduce to 20% one year death probability for ages between 20 and 60
R> for(i in 20:60) itaM2002$qx[itaM2002$x==i]=0.2*itaM2002$qx[itaM2002$x==i]
R> #otbain the reduced mortality table
R> itaM2002reduced<-probs2lifetable(probs=itaM2002[,"qx"], radix=100000,
+                   type="qx",name="IT 2002 Males reduced")
```

An actuarialtable class inherits from the lifecontingencies class, but it contains and additional slot: the interest rate slot,

```
R> #assume 3% interest rate
R> exampleAct=new("actuarialtable",x=exampleLt@x, lx=exampleLt@lx, interest=0.03,
+                   name="example actuarialtable")
```

Method getOmega provides the ω age.

```
R> getOmega(exampleAct)

[1] 9
```

Method print behaves differently between lifetable objects and actuarialtable objects. One year survival probability and complete expected remaining life until deaths is reported when print method is applied on a lifetable object. Classical commutation functions (D_x, N_x, C_x, M_x, R_x) are reported when print method is applied on an actuarialtable object.

```
R> #apply method print applied on a life table
R> print(exampleAct)

Actuarial table  example actuarialtable interest rate  3 %

   x   lx          Dx         Nx        Cx       Mx        Rx
1  0 1000 1000.00000 5467.92787  48.54369 840.7400 4839.7548
2  1  950  922.33010 4467.92787  94.25959 792.1963 3999.0148
3  2  850  801.20652 3545.59778 137.27125 697.9367 3206.8185
4  3  700  640.59916 2744.39125  17.76974 560.6654 2508.8819
5  4  680  604.17119 2103.79209  69.00870 542.8957 1948.2164
6  5  600  517.56527 1499.62090  41.87421 473.8870 1405.3207
7  6  550  460.61634  982.05563 121.96373 432.0128  931.4337
8  7  400  325.23660  521.43929 157.88185 310.0491  499.4210
9  8  200  157.88185  196.20268 114.96251 152.1672  189.3719
10 9   50   38.32084   38.32084  37.20470  37.2047   37.2047

R> #apply method print applied on an actuarial table
R> print(exampleAct)

Actuarial table  example actuarialtable interest rate  3 %

   x   lx          Dx         Nx        Cx       Mx        Rx
1  0 1000 1000.00000 5467.92787  48.54369 840.7400 4839.7548
2  1  950  922.33010 4467.92787  94.25959 792.1963 3999.0148
3  2  850  801.20652 3545.59778 137.27125 697.9367 3206.8185
4  3  700  640.59916 2744.39125  17.76974 560.6654 2508.8819
5  4  680  604.17119 2103.79209  69.00870 542.8957 1948.2164
6  5  600  517.56527 1499.62090  41.87421 473.8870 1405.3207
7  6  550  460.61634  982.05563 121.96373 432.0128  931.4337
8  7  400  325.23660  521.43929 157.88185 310.0491  499.4210
9  8  200  157.88185  196.20268 114.96251 152.1672  189.3719
10 9   50   38.32084   38.32084  37.20470  37.2047   37.2047
```

Basic demographic analysis

Basic demographic estimations can be performed on valid `lifetable` or `actuariatable` objects. Below calculations for ${}_tp_x$, ${}_tq_x$ and $\mathring{e}_{x:\overline{n}|}$.

```
R> #using ips55M life table
R> #probability to survive one year, being at age 20
R> pxt(ips55M,20,1)

[1] 0.9995951

R> #probability to die within two years, being at age 30
R> qxt(ips55M,30,2)
```

```
[1] 0.001332031

R> #expected (curtate) life time between 50 and 70 years
R> exn(ips55M, 50,20)

[1] 19.43322
```

Fractional survival probabilities can also be calculated according with linear interpolation, constant force of mortality and hyperbolic assumption (see Bowers *et al.* (1997) for details).

```
R> #using Society of Actuaries illustrative life table
R> data(soa08Act)
R> pxt(soa08Act,80,0.5,"linear") #linear interpolation (default)

[1] 0.9598496

R> pxt(soa08Act,80,0.5,"constant force") #constant force

[1] 0.9590094

R> pxt(soa08Act,80,0.5,"hyperbolic") #hyperbolic Balducci's assumption.

[1] 0.9581701
```

Calculations of two heads survival probabilities can be performed also, as code below shows:

```
R> #using example life tables
R> pxyt(exampleLt,exampleLt,x=6, y=7, t=2) #joint survival probability

[1] 0.04545455

R> pxyt(exampleLt,exampleLt,x=6, y=7, t=2,status="last") #last survival probability

[1] 0.4431818

R> #using two distinct real life tables (Italian IPS55 tables for males and females)
R> #evaluate the expected joint life time for a couple aged 65 and 63  respectively
R> exyt(ips55M, ips55F, x=65,y=63, status="joint")

[1] 19.1983
```

4.3. Classical actuarial mathematics examples

Classical actuarial mathematics examples on life contingencies are presented. The SOA illustrative life table assuming a 6% interest rates (the same used in most Bowers *et al.* (1997) examples) will be used, unless otherwise stated.

Life insurance examples

Following examples show the APV calculation (i.e. the lump sum benefit premium) for:

1. 10-year term life insurance for a subject aged 30 assuming 4% interest rate, $A^{1}_{30:\overline{10}|}$.
2. 10-year term life insurance for a subject aged 30 with benefit payable at the end of month of death at 4% interest rate.
3. whole life insurance for a subject aged 40 assuming 4% interest rate, A_{40}.
4. 5 years deferred 10-years term life insurance for a subject aged 40 assuming 5% interest rate, ${}_{5|10}\bar{A}_{40}$.
5. 5 years annually decreasing term life insurance for a subject aged 50 assuming 6% interest rate, $(DA)^{1}_{50:\overline{5}|}$.
6. 20 years increasing term life insurance, age 40, $(IA)^{1}_{50:\overline{5}|}$.

```
R> #The APV of a life insurance for a 10-year term life insurance for an
R> #insured aged 40 @ 4% interest rate is
R> Axn(soa08Act, 30,10,i=0.04)

[1] 0.01577283

R> #same as above but payable at the end of month of death
R> Axn(soa08Act, x=30,n=10,i=0.04,k=12)

[1] 0.01605995

R> #a whole life for a 40 years old insured at @4% is
R> Axn(soa08Act, 40) #soa08Act has 6% implicit interest rate

[1] 0.1613242

R> #a 5-year deferred life insurance, 10 years length, 40 years age, @5% interest rate
R> Axn(soa08Act, x=40,n=10,m=5,i=0.05)

[1] 0.03298309

R> #Five years annually decreasing term life insurance, age 50.
R> DAxn(soa08Act, 50,5)

[1] 0.08575918

R> #Increasing 20 years term life insurance, age 40
R> IAxn(soa08Act, 40,10)

[1] 0.1551456
```

while following codes evaluate pure endowments APV, ${}_{n}E_{x}$

```
R> #evaluate the APV for a n year pure endowment, age x=30, n=35, i=6%
R> Exn(soa08Act, x=30, n=35, i=0.06)

[1] 0.1031648

R> #try i=3%
R> Exn(soa08Act, x=30, n=35, i=0.03)

[1] 0.2817954
```

Life annuities examples

Following examples show APV calculations for different annuities variations:

1. annuity immediate for a subject aged 65, a_{65}.
2. annuity due for a subject aged 65, $\ddot{a}_{65}$.
3. 20 years annuity due with monthly fractional payments of \$1000, $\ddot{a}^{(12)}_{65:\overline{20}|}$.

```
R> #annuity immediate
R> axn(soa08Act, x=65, m=1)

[1] 8.896928

R> #annuity due
R> axn(soa08Act, x=65)

[1] 9.896928

R> #due with monthly payments of $1000 provision
R> 12*1000*axn(soa08Act, x=65,k=12)

[1] 113179.1

R> #due with montly payments of $1000 provision, 20 - years term
R> 12*1000*axn(soa08Act, x=65,k=12, n=20)

[1] 108223.5

R> #immediate with monthly payments of 1000 provision, 20 - years term
R> 12*1000*axn(soa08Act, x=65,k=12,n=20,m=1/12)

[1] 107321.1
```

Benefit premiums examples

lifecontingencies package functions can be used to evaluate benefit premium P for life contingencies insurance. A (level) benefit premium is defined as the actuarial present value of the provided coverage paid in h installments, $P = \frac{APV}{\ddot{a}_{x:\overline{h|}}}$.

```
R> #Assume X, aged 30, whishes to buy a $ 250K 35-years life insurance
R> #premium paid annually for 15 years @2.5% interest rate.
R> Pa=100000*Axn(soa08Act, x=30,n=35,i=0.025)/axn(soa08Act, x=30,n=15,i=0.025)
R> Pa

[1] 921.5262

R> #while if the premium is paid on a montly basis the flat benefit premium
R> Pm=100000*Axn(soa08Act, x=30,n=35,i=0.025)/axn(soa08Act, x=30,n=15,i=0.025,k=12)
R> Pm

[1] 932.9836

R> #level semiannual premium for an endowment insurance of 10000
R> #insured age 50, insurance term is 20 years
R> APV=10000*(Axn(soa08Act,50,20)+Exn(soa08Act,50,20))
R> P=APV/axn(soa08Act,50,20,k=2)
R> P

[1] 325.1927
```

Benefit reserves examples

Now we will evaluate the benefit reserve for a 20 year l ife insurance of 100,000, whith benefits payable at the end of year of death, whith level benefit premium payable at the beginning of each year. Assume 3% of interest rate and SOA life table to apply.
The benefit premium is P, determined by equation $P\ddot{a}_{40:\overline{20|}} = 100000A^{\,1}_{40:\overline{20|}}$, while the benefit reserve is determined by equation ${}_kV^{\,1}_{40+t:\overline{n-t|}} = 100000A^{\,1}_{40+t:\overline{20-t|}} - P\ddot{a}_{40+t:\overline{20-t|}}$ for $t = 0 \ldots 19$.

```
R> P=100000*Axn(soa08Act,x=40,n=20,i=0.03)/axn(soa08Act,x=40,n=20,i=0.03)
R> for(t in 0:19) cat("At time ",t," benefit reserve is ", 100000*Axn(soa08Act,x=40+t,n=20

At time  0  benefit reserve is  0
At time  1  benefit reserve is  306.9663
At time  2  benefit reserve is  604.0289
At time  3  benefit reserve is  889.0652
At time  4  benefit reserve is  1159.693
At time  5  benefit reserve is  1413.253
At time  6  benefit reserve is  1646.808
```

```
At time  7  benefit reserve is  1857.044
At time  8  benefit reserve is  2040.286
At time  9  benefit reserve is  2192.436
At time  10  benefit reserve is  2308.88
At time  11  benefit reserve is  2384.513
At time  12  benefit reserve is  2413.576
At time  13  benefit reserve is  2389.633
At time  14  benefit reserve is  2305.464
At time  15  benefit reserve is  2152.963
At time  16  benefit reserve is  1922.973
At time  17  benefit reserve is  1605.162
At time  18  benefit reserve is  1187.872
At time  19  benefit reserve is  657.8482
```

The benefit reserve for a whole life annuity with level annual premium is ${}_kV({}_{n|}\ddot{a}_x)$, that equals ${}_{n|}\ddot{a}_x - \bar{P}({}_{n|}\bar{a}_x)\ddot{a}_{x+k:\overline{n-k|}}$ when $x \dots n$, $\ddot{a}_{x+k}$ otherwise. The figure is shown in 1.

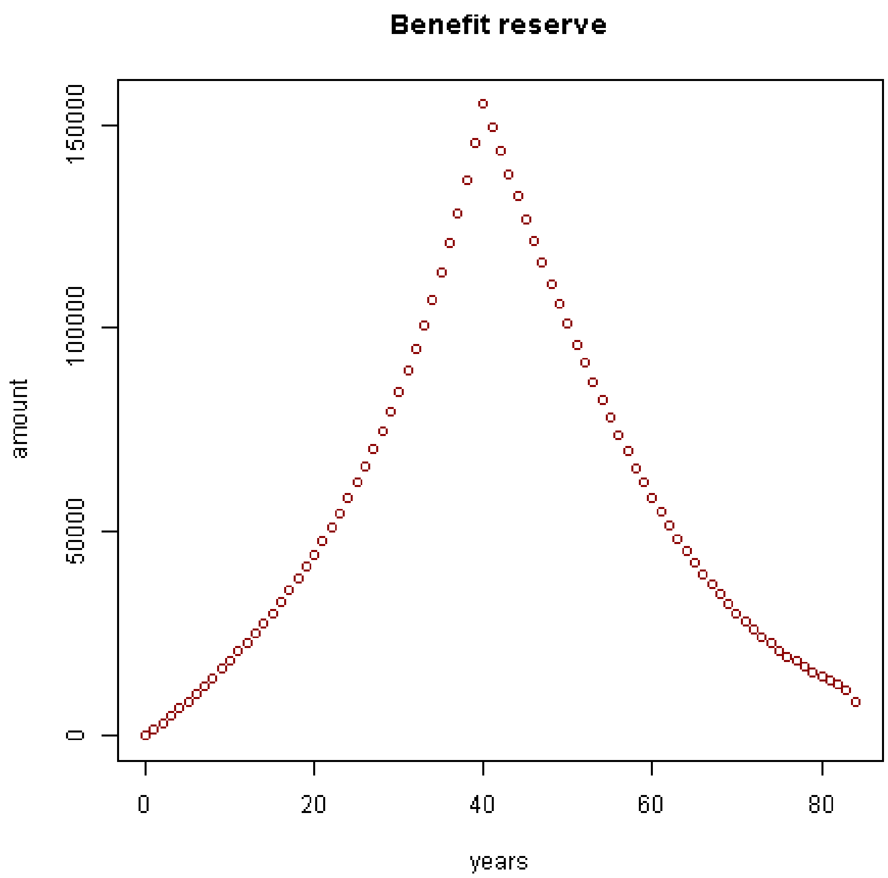

Figure 1: Benefit reserve of $\ddot{a}_{65}$

Insurance and annuities on two heads

Lifecontingencies package provides functions to evaluate life insurance and annuities on two lifes. Following examples will check the equality $a_{\overline{xy}} = a_x + a_y - a_{xy}$.

```
R> axn(soa08Act, x=65,m=1)+axn(soa08Act, x=70,m=1)- axyn(soa08Act,soa08Act, x=65,y=70,stat
```

```
[1] 10.35704
```

```
R> axyn(soa08Act,soa08Act, x=65,y=70, status="last",m=1)
```

```
[1] 10.35704
```

Reversionary annuities (annuities payable to life y upon death of x), $a_{x|y} = a_y - a_{xy}$ can also be evaluate combining **lifecontingencies** functions.

```
R> #assume x aged 65, y aged 60
R> axn(soa08Act, x=60,m=1)-axyn(soa08Act,soa08Act, x=65,y=60,status="joint",m=1)
```

```
[1] 2.695232
```

4.4. Stochastic analysis

This last paragraphs will show some stochastic analysis that can be performed by our package, both in demographic analysis and life insurance evaluation.

Demographic examples

The age-until-death, both in the continuous (T_x) or curtate form (K_x), is a stochastic variable whose distribution is implicit within the deaths tabulated within a life table. The code below shows how to sample values from the age-until-death distribution implicit in the SOA life table.

```
R> data(soa08Act)
R> #sample 10 numbers from the Tx distribution
R> sample1<-rLife(n=10,object=soa08Act,x=0,type="Tx")
R> #sample 10 numbers from the Kx distribution
R> sample1<-rLife(n=10,object=soa08Act,x=0,type="Kx")
```

while code below shows how the mean of the sampled distribution is statistically equivalent to the expected life time.

```
R> #assume an insured aged 29
R> #his expected integer number of years until death is
R> exn(soa08Act, x=29,type="curtate")

[1] 45.50066

R> #check if we are sampling from a statistically equivalent distribution
R> t.test(x=rLife(2000,soa08Act, x=29,type="Kx"),mu=exn(soa08Act, x=29,type="curtate"))$p.

[1] 0.8184837

R> #statistically not significant
```

Finally figure 2 shows the deaths distribution implicit in the ips55M life table.

Actuarial mathematics examples

The APV is a present value of a random variable that represents a composite function between the discount amount and indicator variables regarding the life status of the insured. This is the present value of benefits random variable, $\tilde{Z}$.
Usually life contingencies evaluation functions return the APV, as the `type` parameter has "EV" (expected value) as default value. However most life contingencies insurance evaluation functions are provided with a "ST" parameter that allows to obtain a variate from the underlying present value benefit distribution.
We will apply these concepts to term life insurance with code below:

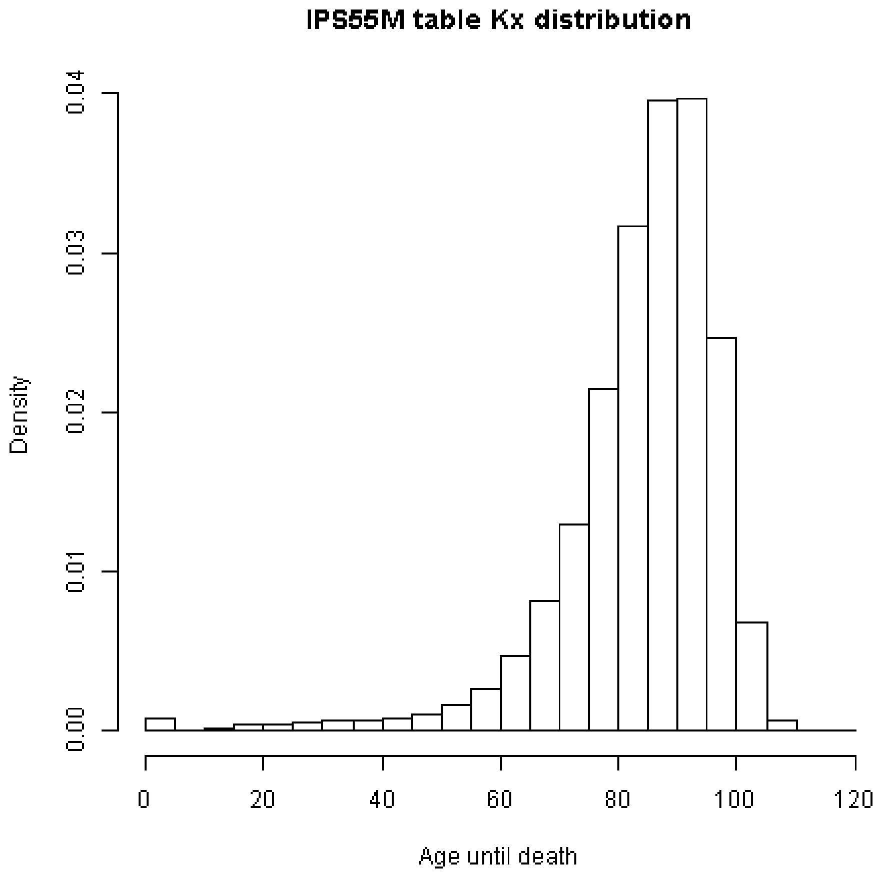

Figure 2: Deaths distribution implicit in the IPS55 males table

```
R> #term life insurance 25 years old, 40 year term
R> #APV
R> APV=Axn(soa08Act,x=25,n=40,type="EV")
R> APV

[1] 0.0479709

R> #analysis of the distribution
R> sampleAxn=numeric(1000)
R> for(i in 1:1000) sampleAxn[i]=Axn(soa08Act,x=25,n=40,type="ST")
R> summary(sampleAxn)

   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
0.00000 0.00000 0.00000 0.04775 0.00000 0.94340

R> t.test(x=sampleAxn,mu=APV)$p.value

[1] 0.9526489
```

```
R> #unbiased
```

to increasing life insurance

```
R> #increasing life insurance 25 years old, 40 year term
R> #APV
R> APV=IAxn(soa08Act,x=25,n=40,type="EV")
R> APV

[1] 1.045507

R> #analysis of the distribution
R> sampleIAxn=numeric(1000)
R> for(i in 1:1000) sampleIAxn[i]=IAxn(soa08Act,x=25,n=40,type="ST")
R> summary(sampleIAxn)

   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  0.000   0.000   0.000   1.137   0.000   6.313

R> #verify if sampleIAxn expected value is APV
R> t.test(x=sampleIAxn,mu=APV)$p.value

[1] 0.1757657

R> #unbiased
```

Current **lifecontingencies** allows to obtain stochastic sample of present value benefits function for all the one head life contingencies insurances. `rLifeContingencies` function allow to obtain a sample of size n from any life contingency insurance.

```
R> #sample from the distribution of an annuity due
R> #x=65
R> simAnnuity<-rLifeContingencies(n=1000, lifecontingency="axn",object=soa08Act,
+  x=65,t=getOmega(soa08Act)-65, i=soa08Act@interest,m=0,k=1)
```

Figure 3 shows the distribution of simAnnuity vector, whose APV is $\ddot{a}_{65}$.

5. Discussion

The **lifecontingencies** package allows actuaries to perform financial mathematics and life contingencies actuarial mathematics within R. It offers the basic tools to manipulate life tables and perform financial calculations. Pricing, reserving and stochastic evaluations of most important life insurance contract can be performed within R.

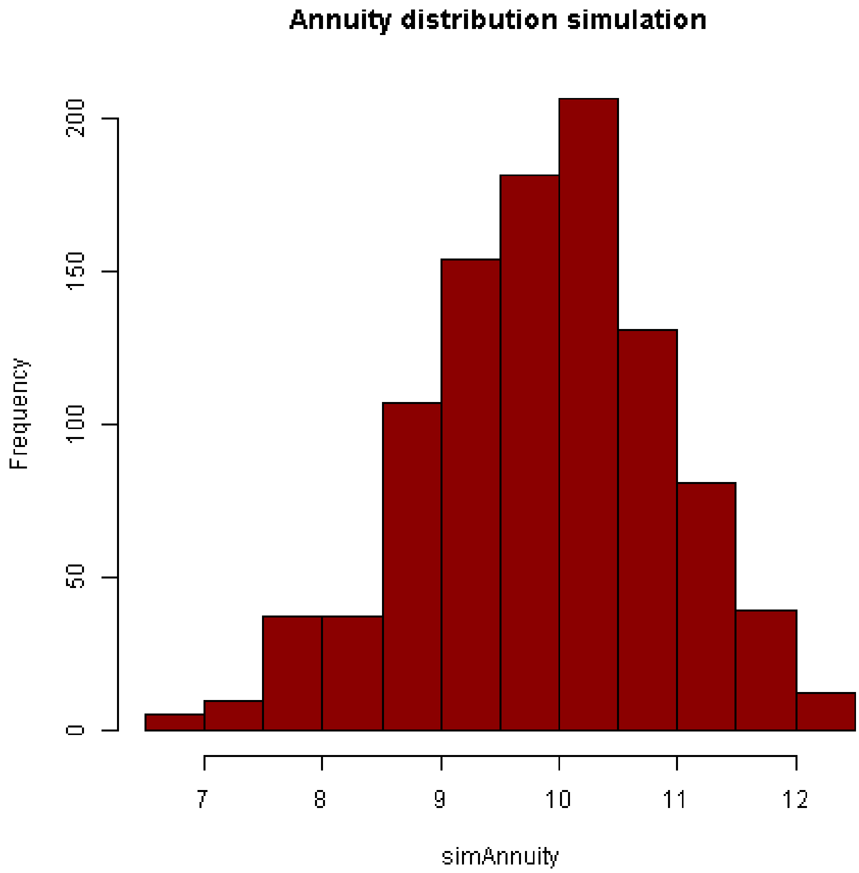

Figure 3: Stochastic distribution of $\ddot{a}_{65}$

One of the most important limitations of **lifecontingencies** is handling only single decrement tables. In the future the `lifetable` class will probably be expanded to handle multiple decrement causes. Moreover in the future we expect to to provide coerce methods toward packages specialized in demographic analysis, like **demography** and **LifeTables**. Communciation with interest rates modelling packages, as **termstrcR** will be also explored.

Disclaimer

The accuracy of calculation have been verified by checkings with numerical examples reported in Bowers *et al.* (1997). The package numerical results are identical to those reported in the Bowers *et al.* (1997) for most function, with the exception of fractional payments annuities where the accuracy leads only to the 5th decimal. The reason of such inaccuracy is due to the fact that the package calculates the APV by directly sum of fractional survival probabilities, while the formulas reported in Bowers *et al.* (1997) uses an analytical formula.

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I wish to thank Christophe Dutang and Tim Riffle for their valuable suggestions.

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