Stationarity Tests and A New Prediction Method for Functional Time Series

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Abstract

ftsa 4.7 enhances the previous versions by adding tests for stationarity of functional time series and another method for their prediction. A new data set of particulate pollution curves has been added.

Keywords: stationary test, functional time series forecasting, VARMA.

1. Testing stationarity of functional time series

A functional time series is a sequence of curves $X_1(t), X_2(t), \ldots, X_N(t)$. Each curve is defined on the same grid of points in an interval $[T_1, T_2]$. We use N to denote the size of the sample of functions because t is used as the argument of the functions, i.e. a point in the interval $[T_1, T_2]$. In Functional Data Analysis, each function X_i is viewed as an element of a function space, the most general of such spaces is the space of square integrable functions, denoted $L^2 = L^2([T_1, T_2])$ (see e.g., Horváth and Kokoszka 2012, Chapter 2). Just like for scalar and vector time series, many procedures developed for functional time series require that these series be stationary. Function `T_stationary` implements two tests of stationarity developed by Horváth, Kokoszka, and Rice (2014). The null hypothesis of these tests is that the series is stationary, in the strict sense, i.e. that for any h and any n

$$(X_{1+h}, X_{2+h}, \ldots, X_{n+h}) \stackrel{d}{=} (X_1, X_2, \ldots, X_n), \tag{1}$$

where the equality in distribution refers to the equality of distributions in the product space $L^2 \times L^2 \times \ldots \times L^2$ (n times). The alternative hypothesis is that the sequence of functions X_i is not stationary in the above sense. For example, it can contain change points, trends, or random walk components. We emphasize, that each function $t \mapsto X_i(t)$ is typically a realization of a continuous time nonstationary process observed at some discrete points t_j. The stationarity refers to the sequence of functions $X_1, X_2, \ldots, X_N$. This point is illustrated in Figure 1. The top panel shows price curves on five consecutive days. Each of these curves can be denoted X_i, and only $N = 5$ curves are shown. In typical applications, N is much larger, from several dozen to several thousand. The sequence of price curves is in general not stationary. Even for the five displayed curves an upward trend is seen, such a trending or random walk behavior is much more pronounced for longer series. The Bottom panel of Figure 1 shows the same curves, but suitably normalized. Even though each curve is a realization of a nonstationary stochastic process, such normalized curves, known as cumulative intraday return curves (see Kokoszka, Miao, and Zhang 2015, and references therein), form a stationary functional time series.

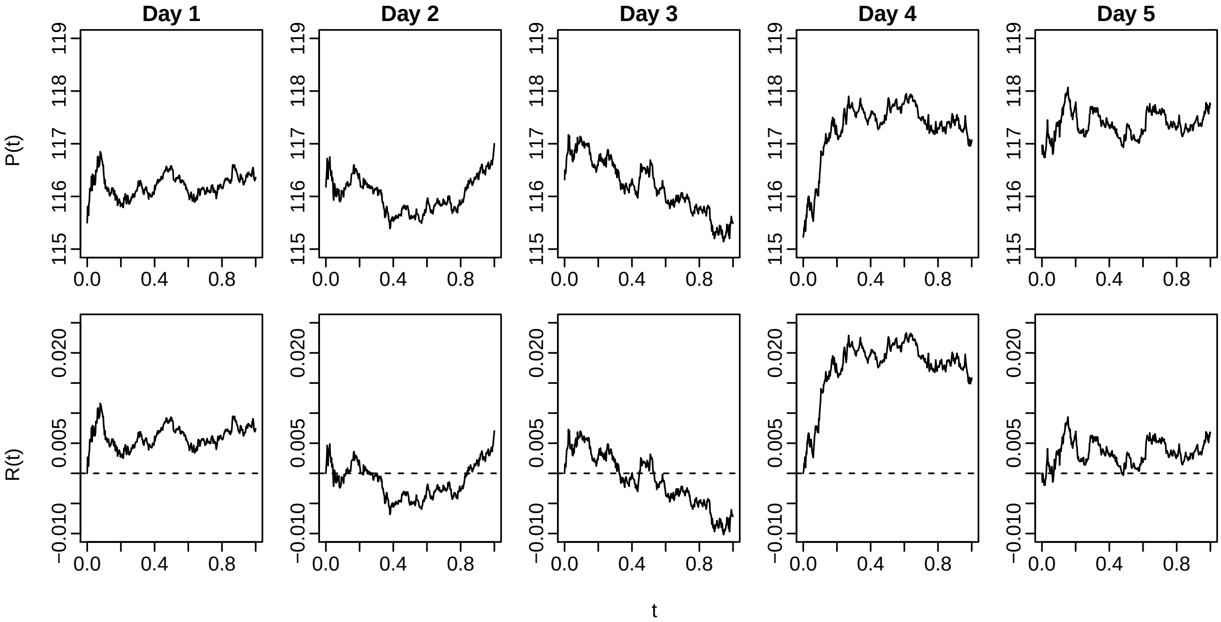


Figure 1: Top: IBM price curves on five consecutive trading days. Bottom: Cumulative intraday returns on these prices.

We illustrate the application of the tests using the functional time series `pm_10_GR_sqrt` which has been added to **ftsa** 4.7 (Hyndman and Shang 2015). The data set `pm_10_GR` contains half–hourly measurements of the concentration of particulate matter less than 10um (pm10) in Graz, Austria from October 1, 2010 to March 31, 2011. This is a functional time series with $N = 182$ daily curves. To stabilize variability of these functions, square root of each half–hourly observation is computed; the functional time series so transformed is available as `pm_10_GR_sqrt`. The call

```
require(ftsa)
T_stationary(pm_10_GR_sqrt$y)
```

produces the following output

```
Monte Carlo test of stationarity of a functional time series
null hypothesis: the series is stationary
p-value = 0.082
N (number of functions) = 182
number of MC replications = 1000
```

The p–value of 8.2% indicates that the series can be treated as stationary. Since the p-value is less than 10%, using a larger sample size might reveal some nonstationarity due to seasonal effects. The null distribution of this test does not have a closed form; it must be approximated by a Monte Carlo distribution. The last line indicates that one thousand replications were used, the default value. The call

```
T_stationary(pm_10_GR_sqrt$y, J = 100, MC_rep = 5000, h = 20, pivotal = TRUE)
```

produces the output

```
Pivotal test of stationarity for a functional time series
null hypothesis: the series is stationary
p-value = 0.1188
N (number of functions) = 182
number of MC replications = 5000
```

The main difference relative to the previous call is that the argument `pivotal = TRUE` indicating that a different tests statistic is used, which has a pivotal asymptotic distribution. Nevertheless, the distribution of the test statistic is still approximated by a Monte Carlo distribution to ensure a more accurate empirical size. If `pivotal = FALSE`, the statistic T_N defined in Section 2.1 of Horváth *et al.* (2014) is used; if `pivotal = TRUE`, their statistic $T_N^0(d)$ defined in Section 2.2 is used. The argument `J` is the truncation level used to approximate the limit distribution defined by an infinite series, only the first J terms of this series are used. The distribution so truncated is used in Monte Carlo replications. The argument h is the kernel bandwidth. Both are defined in Section 4.1 of Horváth *et al.* (2014). The help file of the function `T_stationary` explains other tuning parameters used in that paper, which are provided as arguments.

2. Forecasting functional time series

The package **ftsa** 4.7 implements a new method of predicting functional time series proposed by Aue, Norinho, and Hörmann (2015). The main difference between the new forecasting method implemented in `farforecast` and the existing method implemented in `forecast.ftsm` is as follows. In both methods, the functions X_i are represented as

$$X_i(t) \approx \mu(t) + \sum_{j=1}^{J} \xi_{ij}\hat{v}_j(t), \tag{2}$$

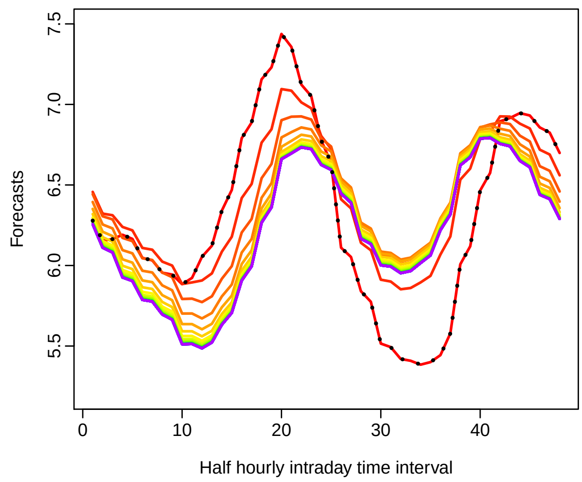
where the $\hat{v}_j$ are the estimated functional principal components, EFPC, (see e.g., Horváth and Kokoszka 2012, Chapter 3). The function `forecast.ftsm` treats each series $\xi_{1j}, \xi_{2j}, \xi_{3j}, \dots, \xi_{Nj}$ as a univariate time series and computes the predictions of its future values using the automatic autoregressive integrated moving average algorithm of Hyndman and Khandakar (2008). These predictions are used to construct the predicted curves using the EFPC decomposition above. The function `farforecast` treats the scores as a J-dimensional time series $[\xi_{i1}, \xi_{i2}, \dots \xi_{iJ}]$, $i = 1, 2, 3, \dots, N$, and applies a multivariate prediction algorithm assuming that this series is a stationary vector autoregression. Before using `farforecast` it is therefore advisable to transform the original functional time series to a stationary series and verify stationarity using the function `T_stationary`. Other arguments of the function `farforecast` are analogous to the arguments of `forecast.ftsm`, and are explained in the help file.

In Figure 2, we compare the differences between multivariate and univariate time-series forecasting algorithms for predicting one-day-ahead and 30-days-ahead pm10 pollution curves. The following code produces Figure 2.

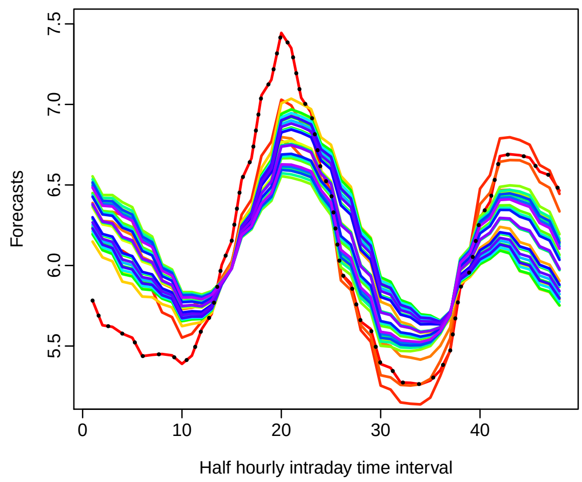
```
# Multivariate time-series prediction algorithm
require(ftsa); require(vars)
```

```
h30_forecast_sqrt_pm10 = farforecast(ftsm(pm_10_GR_sqrt), h = 30, PI = FALSE)
plot(h30_forecast_sqrt_pm10, ylim = c(5.2,7.5),
     xlab = "Half hourly intraday time interval",
     ylab = "Forecasts", lwd = 2)
h1_forecast_sqrt_pm10 = farforecast(ftsm(pm_10_GR_sqrt), h = 1)
lines(h1_forecast_sqrt_pm10, lwd = 3, lty = 3)

# Univariate time-series prediction algorithm
h30_forecast_sqrt_pm10_ftsm = forecast(ftsm(pm_10_GR_sqrt), h = 30, method = "arima")
plot(h30_forecast_sqrt_pm10_ftsm, ylim = c(5.2,7.5), lwd = 2)
h1_forecast_sqrt_pm10_ftsm = forecast(ftsm(pm_10_GR_sqrt), h = 1, method = "arima")
lines(h1_forecast_sqrt_pm10_ftsm$mean, lwd = 3, lty = 3)
```



(a) Multivariate time-series forecasting algorithm



(b) Univariate time-series forecasting algorithm

Figure 2: Predicted 30–days–ahead pm10 pollution curves; one–day–ahead prediction highlighted with black dots.

3. Conclusion

This article describes a method in the **ftsa** package for testing the stationarity of a functional time series. For a stationary functional time series, a new prediction method is introduced by forecasting principal component scores via a multivariate time-series forecasting method. Both procedures are illustrated by an application to the concentration of particulate matter data set. We used the test to verify its stationarity, and produced one-step-ahead and 30-steps-ahead forecasts via the new prediction method. To sum up, these two additional methods should be considered when the interest lies in forecasting future realizations of a time series of functions. The test should be used before applying any inferential tools that assume stationarity.

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