

# The `fastclime` Package for Linear Programming and Constrained $l_1$ -Minimization Approach to Sparse Precision Matrix Estimation in $\mathbf{R}$

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April 28, 2014

## Abstract

We describe an  $\mathbf{R}$  package called `fastclime`, which provides a generic linear programming solver and a parameterized linear programming solver. It also contains a solver for an important sparse precision matrix estimator named CLIME. Our package efficiently implements a method called *the parametric simplex method*. Compared with the existing package `clime`, `fastclime` has many extra advantages: (1) it gives a full piece-wise linear regularization solution path of the CLIME tuning parameter for each column; (2) it only requires a few iterations to recover each sparse precision matrix column; (3) the entire solver is written in C and is easy to modify.

## 1 The Parametric Simplex Method

In this section, we will follow the normal linear programming convention and use  $x$  as the normal linear programming variable which has dimension  $n$ . Here we briefly describe the primal simplex method and introduces the parametric simplex methods.

We give the standard linear programming in inequality form:

$$\max c^T x \quad \text{subject to: } Ax \leq b, \quad x \geq 0 \quad x \in \mathbb{R}^n \quad (1)$$

where  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$ ,  $c \in \mathbb{R}^n$  are given.

For the primal simplex method, we require that  $b \geq 0$ . This property is called *primal feasibility*. We change the standard inequality form into equality form by adding *slack variables*  $w$ :

$$\max [c \ 0] \begin{bmatrix} x \\ w \end{bmatrix} \quad \text{subject to: } [A \ I] \begin{bmatrix} x \\ w \end{bmatrix} = b \quad x, w \geq 0 \quad x \in \mathbb{R}^n \quad w \in \mathbb{R}^m \quad (2)$$

For notation convenience, we rewrite the variable  $x$  as:

$$x = [x_1 \ x_2 \ \cdots \ x_n \ w_1 \ \cdots \ w_m]^T = [x_1 \ x_2 \ \cdots \ x_n \ x_{n+1} \ \cdots \ x_{n+m}]^T$$

The new variables,  $x_{n+1}, \dots, x_{n+m}$ , are called *slack variables*. We separate the  $m + n$  variables into two parts: *basic variables* and *nonbasic variables*. Basics variables are used to represent the nonbasic variables. Initially, the slack variables are basic variables and the original variables are nonbasic variables. We separate the matrix  $[A \ I]$  into the basic part and the nonbasic part as well (with permutation). Initially the matrix  $A$  corresponds to nonbasic variables and the identity matrix part corresponds to basic variables. The vector  $[c \ 0]$  is separated in the same fashion:

$$[A \ I] = [N \ B] \quad \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} x_N \\ x_B \end{bmatrix} \quad \begin{bmatrix} c \\ 0 \end{bmatrix} = \begin{bmatrix} c_N \\ c_B \end{bmatrix}$$

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Now the constraints of (2) will become:  $Nx_N + Bx_B = b$ . We denote the objective  $c^T x$  as  $\zeta$ , then we have:

$$\begin{aligned} x_B &= B^{-1}b - B^{-1}Nx_N \\ &= x_B^* - B^{-1}Nx_N \end{aligned}$$

$$\begin{aligned} \zeta &= c^T x \\ &= c_B^T x_B + c_N^T x_N \end{aligned} \tag{3}$$

$$\begin{aligned} &= c_B^T (B^{-1}b - B^{-1}Nx_N) + c_N^T x_N \\ &= c_B^T B^{-1}b - ((B^{-1}N)^T c_B - c_N)^T x_N \\ &= \zeta^* - (z_N^*)^T x_N \end{aligned} \tag{4}$$

with  $\zeta^* = c_B^T B^{-1}b$ ,  $x_B^* = B^{-1}b$  and  $z_N^* = (B^{-1}N)^T c_B - c_N$ .

We call equations (3) and (4) the *primal dictionary* associated with the current basis  $B$ . Primal feasibility requires that  $x_B \geq 0$ , which is initially guaranteed by  $b \geq 0$ . We read off the specific values of  $x_B$  and  $\zeta$  by setting  $x_N$  to be zero. For next iteration, we swap one variable from the basics part and one variable from the nonbasic part, and then we write down the dictionary and read off the values such that the objective value is always increasing.

The dual of the problem (1) is:

$$\min b^T y \quad \text{subject to: } A^T y \geq c, \quad y \geq 0 \quad y \in \mathbb{R}^m \tag{5}$$

As with the primal problem, We form by introducing slack variables here as well:

$$\min b^T y \quad \text{subject to: } A^T y - z = c, \quad y, z \geq 0 \quad y \in \mathbb{R}^m \quad z \in \mathbb{R}^n \tag{6}$$

The dual variables are given by:

$$z = [z_1 \quad z_2 \quad \cdots \quad z_n \quad y_1 \quad \cdots \quad y_m]^T = [z_1 \quad z_2 \quad \cdots \quad z_n \quad z_{n+1} \quad \cdots \quad z_{n+m}]^T$$

Similar to (3) and (4), the corresponding *dual dictionary* is given by:

$$z_N = (B^{-1}N)^T c_B - c_N + (B^{-1}N)^T z_B = z_N^* + (B^{-1}N)^T z_B \tag{7}$$

$$-\xi = -c_B^T B^{-1}b + (B^{-1}b)^T z_B = -\zeta^* - (x_B^*)^T z_B \tag{8}$$

where  $\xi$  denotes the objective function in the dual form.

For each dictionary, we set  $x_N$  and  $z_B$  to 0 (complementarity) and read off the solutions to  $x_B$  and  $z_N$  according to (3) and (7). Next, we update the dictionary by removing one basic index and replacing it with a nonbasic index, then we get an updated dictionary. The simplex method produces a sequence of steps to *adjacent* bases such that the value of the objective function is always increasing at each step. Primal feasibility requires that  $x_B \geq 0$ , so while we update the dictionary, primal feasibility must always be satisfied. This process will stop when  $z_N \geq 0$  (dual feasibility), and this is the optimality condition since it satisfies primal feasibility, dual feasibility and complementarity.

Now we introduce the parametric simplex method [Vanderbei, 2008]. Generally speaking, we cannot make the initial primal feasible assumption ( $b \geq 0$ ). The method we use here is to add some nonnegative perturbation times a positive parameter  $\lambda$  to both objective function ( $\bar{c}$ ) and the right hand side of the primal problem ( $\bar{b}$ ). Now (1) becomes:

$$\max (c + \lambda \bar{c})^T x \quad \text{subject to: } Ax \leq b + \lambda \bar{b}, \quad x \geq 0 \quad x \in \mathbb{R}^n \tag{9}$$

The dictionary of equations (3), (4), (7) and (8) will become:

$$x_B = (x_B^* + \lambda \bar{x}_B) - B^{-1}Nx_N \tag{10}$$

$$\zeta = \zeta^* - (z_N^* + \lambda \bar{z}_N)^T x_N \quad (11)$$

$$z_N = (z_N^* + \lambda \bar{z}_N) + (B^{-1}N)^T z_B \quad (12)$$

$$-\xi = -\zeta^* - (x_B^* + \lambda \bar{x}_B)^T z_B \quad (13)$$

with  $\bar{x}_B = B^{-1}\bar{b}$  and  $\bar{z}_N = (B^{-1}N)^T \bar{c}_B - \bar{c}_N$ .

we choose  $\bar{b}$  and  $\bar{c}$ , so that when  $\lambda$  is large, the dictionary will be both primal and dual feasible ( $x_B^* + \lambda \bar{x}_B \geq 0$  and  $z_N^* + \lambda \bar{z}_N \geq 0$ ). At this point, we start to decrease  $\lambda$ . The smallest value of  $\lambda$  is given by

$$\lambda^* = \min\{\lambda : z_N^* + \lambda \bar{z}_N \geq 0 \text{ and } x_B^* + \lambda \bar{x}_B \geq 0\} \quad (14)$$

We interchange one basic variable and one nonbasic variable and update the dictionary in a way so that the value of  $\lambda$  will be decreased. Eventually the value of  $\lambda$  will be decreased to zero which corresponds to the original problem. For each update, we need to make sure that the primal and dual feasibility is satisfied. The detail of one iteration of the parametric simplex method can be found in page 119-121 of [Vanderbei, 2008]. This algorithm must stop in finite time because: (1) the value of  $\lambda$  is decreasing for each iteration; (2) the optimal value corresponding to  $\lambda$  is 0 (3) there are finitely many bases and it indicates that there are only finitely many steps (unless it cycles). The linear programming solver based on the algorithm described above is contained in our package.

In section 2, we will introduce CLIME and apply the parametric simplex method to CLIME. In section 3, we will introduce the design of our software and provide some examples. In section 4, we will show numerical benchmarks and comparisons with other implementations.

## 2 CLIME

In this section, CLIME will be discussed and the normal statistical notation will be used. Estimating the covariance matrix and its inverse is an important and widely discussed problem in statistics. Let  $X = (X_1, \dots, X_d)^T$  be a  $d$ -variate random vector with Gaussian distribution  $N(\mu, \Sigma)$ . Let  $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^d$  be  $n$  observations of a  $d$ -dimensional random vector  $\mathbf{X} = (X_1, \dots, X_d)^T$ . We denote the covariance matrix  $\Sigma = \mathbb{E}\mathbf{X}\mathbf{X}^T$ . It can be easily estimated by its sample covariance:

$$\Sigma_n = \frac{1}{n} \sum_{k=1}^n (x_k - \bar{x})(x_k - \bar{x})^T \quad (15)$$

where  $\bar{x} = \frac{1}{n} \sum_{k=1}^n x_k$  is the sample mean.

The precision matrix, sometimes called the concentration matrix,  $\Omega$ , is defined as  $\Omega = \Sigma^{-1}$ . Under Gaussian model,  $\Omega_{jk} = 0$  indicates that  $X_i$  and  $X_j$  are independent given all other variables  $X_k (k \neq i, j)$ . A closely related problem is the selection of graphical model. Let a graph  $G = (V, E)$  denote the conditional independence relation between the variables  $X$ . More specifically, the vertex set  $V$  corresponds to the set of variables  $X_1, X_2, \dots, X_p$ . An edge between node  $i$  and node  $j$  belongs to the edge set  $E$  if and only if the two variables  $X_i$  and  $X_j$  is not independent given  $X_k (k \neq i, j)$ . Therefore, for Gaussian distribution, recovering the structure of the conditional independence graph is equivalent to estimating the support of the corresponding precision matrix.

Unlike for the covariance matrix, there is no direct method to estimate the precision matrix. Our goal is to estimate the sparse precision matrix when the dimension  $d$  is large. One famous method to estimate the precision matrix is called  $l_1$  regularized log-determinant program [Banerjee et al., 2008]:

$$\Omega = \operatorname{argmin} \operatorname{trace}(\Sigma_n X) - \log(\det(X)) + \lambda \|X\|_1 \text{ subject to: } X \succ 0 \quad (16)$$

This estimator can be solved efficiently by methods such as GLASSO (Graphical Lasso)[Friedman et al., 2007] [Friedman et al., 2010].

Another simple and important linear programming estimator proposed recently is called CLIME (Constrained  $l_1$  Minimization Estimator). Cai et al. define the CLIME estimator  $\hat{\Omega}$  as the solution to the following problem:

$$\min \|\Omega\|_1 \quad \text{subject to: } |\Sigma_n \Omega - I_d|_{max} \leq \lambda, \quad \Omega \in \mathbb{R}^{d \times d} \quad (17)$$

where  $\lambda$  is defined as a tuning parameter,  $\Sigma_n$  is the sample covariance matrix and  $I_d$  is the  $d$ -dimensional identity matrix. This minimization problem can be further decomposed into  $d$  smaller problems by simply recovering the precision matrix column by column:

$$\min \|\beta\|_1 \quad \text{subject to: } |\Sigma_n \beta - e_i|_\infty \leq \lambda \quad \beta \in \mathbb{R}^d \quad (18)$$

where  $\|\beta\|_1 = \sum_{j=1}^d |\beta_j|$  and  $e_i \in \mathbb{R}^d$  is the  $i$ -th unit vector.

Of course, the estimator we obtain by solving these series of linear programming problems are not symmetric, but we can simply take the smaller value of  $\hat{\Omega}_{ij}$  and  $\hat{\Omega}_{ji}$  as the value chosen by the estimator. The result can be shown to be positive definite with high probability. The error between the estimator  $\hat{\Omega}$  and  $\Omega$  satisfies  $\|\hat{\Omega} - \Omega\|_2 = O_p(s\sqrt{\log(d)/n})$  and  $|\hat{\Omega} - \Omega|_\infty = O_p(\sqrt{\log(d)/n})$ , where  $s$  denotes the sparsity of the precision matrix. Moreover, it has been shown numerically that CLIME method uniformly outperforms GLASSO [Cai et al.]. This is the reason why CLIME is very important and useful, although GLASSO and many other maximum likelihood methods are faster compared with linear programming approach.

Now we are ready to apply the parametric simplex method to solve CLIME. By setting  $\beta = \beta^+ - \beta^-$  and  $\|\beta\|_1 = \beta^+ + \beta^-$  and let  $\beta^+ \geq 0$   $\beta^- \geq 0$ , and one of  $\beta^+$ ,  $\beta^-$  be 0, equation (18) of CLIME can be written in the following form:

$$\min \beta^+ + \beta^- \quad \text{subject to: } \begin{pmatrix} \Sigma_n & -\Sigma_n \\ -\Sigma_n & \Sigma_n \end{pmatrix} \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix} \leq \begin{pmatrix} \lambda + e_i \\ \lambda - e_i \end{pmatrix} \quad (19)$$

Equation (19) is clearly an initial setup of a parametric simplex problem. Comparing it with (9), we have the following identifications:

$$A = \begin{pmatrix} \Sigma_n & -\Sigma_n \\ -\Sigma_n & \Sigma_n \end{pmatrix} \quad b = \begin{pmatrix} e_i \\ -e_i \end{pmatrix} \quad c = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} \quad \bar{c} = 0 \quad \bar{b} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

By constructing this setup and plugging the problem into our parametric simple linear programming solver, we are able to recover the full piecewise-linear regularization path in a relatively short time.

### 3 Design, Implementation and Examples

The package `fastclime` consists of two major parts: a pair of linear programming solvers and a series of functions used to estimate precision matrix: data generator, graph estimation, graph visualization.

For the data generate part, the function `fastclime.generator()` is able to generate multivariate Gaussian data with different graph structures.

`fastclime` is the main function used to estimate the graph. It takes three parameters. The first parameter can be either a data matrix or a sample covariance matrix. The second parameter is the smallest value of  $\lambda$  the user would like the solver to explorer. The third parameter is the designed iteration numbers required for each column (the length of the solution path). The function estimates the precision matrix column by column, and stops when either the smallest value of  $\lambda$  has been reached at each column or the maximum designed iteration number has been reached. The estimator is based on a parametric simplex linear programming solver written in C. In order to maintain the speed of the program, the default path length is set to be 50. The user must be very careful when asking for a large solution path when the dimension  $d$  is large. This estimator is designed to take advantage of the parametric simplex method to recover sparse

precision matrix only. When the precision matrix is not sparse, it will take a long time to recover and the result is not that meaningful.

The output object of the main function has several components. The list of precision matrices is stored in *icovlist*. *lambdamtx* includes the full path information for every column, with zero filled in when the function finds the required minimum value of  $\lambda$  in that column. *maxnlambda* is the length of the solution path. If the function finds the required minimum value of  $\lambda$  in less than *nlambda* iterations for all columns, then the actual maximum length for all columns will be returned; otherwise it equals the input *nlambda*.

`fastclime.lambda` is the selector function in this package. It selects the precision matrix and solution path for a given parameter  $\lambda$ . The first two inputs take the output from `fastclime` directly: *icovlist* and *lambdamtx*. The third parameter is user specified parameter *lambda*. The function is able to estimate the precision matrices corresponding to all  $\lambda$  values ranging from 1 to *lambda.min*, as long as a large *nlambda* is used `fastclime`.

The plotting functions `fastclime.plot()` provides a method to plot the undirected graph at each path. Please note only the adjacency matrix is allowed for this function.

We illustrate the user interface by two examples. The first one is based on the data generated by `fastclime.generator()`,

```
> library(fastclime) # Load the package
> L = fastclime.generator(n=200,d=50,graph="hub") # Generate data with hub structures
> out = fastclime(L$data,0.1) # Estimate the solution path
> plot(out)
# Plot the sparsity level information from the graph path
> out1 = fastclime.lambda(out$lambdamtx, out$icovlist, 0.2)
# Finds the estimated path corresponding to lambda=0.2
```

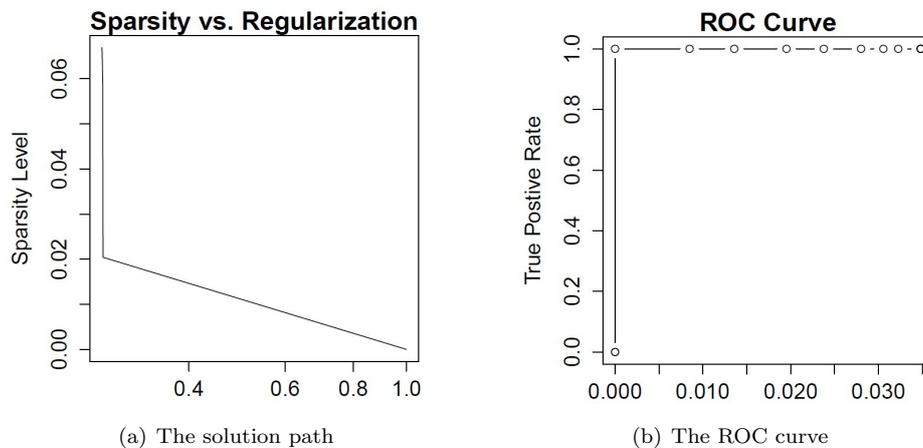


Figure 1: The solution path and the ROC curve of the first example

In this example, we first generate 200 samples from a 50-dimensional Gaussian distribution with hub structure. All information, such as the true covariance matrix, the sample covariance matrix, the true precision matrix and the adjacency matrix are stored in object *L*. We try to estimate the inverse covariance matrix by `fastclime`, and we assume the sparsity level is about 0.1, which is the default value. The estimator we obtain is stored in an object called *out*. We observe the result has a path length of 14. The ROC curve and the solution path is shown in Figure 1(a) and Figure 1(b), respectively. As shown in Figure 1(a), this method gives a almost perfect ROC curve. The true sparsity level is 0.06. After three iterations, it recovers all the tree correlation between the variables. The user has to be aware that as the path length increases, it is likely to obtain additional undirected edges in the graph, as shown in Figure 2(c).

The second example is based on some stock market data which is contained in the package. The data contains stock price from S&P 500 during the period between Jan 1st, 2003 and Jan 1st, 2008. It gives 1258 samples (*n*) from 452 stocks (*d*). We give a approximate sparsity ratio to be

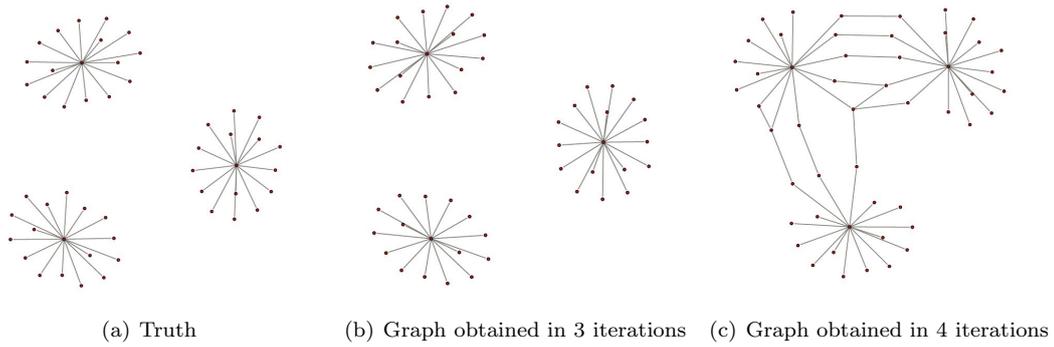


Figure 2: Graphs estimated by fastclime

0.05. The program automatically calculate the solution path. We reach the sparsity level 0.05 after 30 iterations. The solution path for the first column is shown in Figure 3, and a few example of estimated graphs with corresponding sparsity level labeled are shown in Figure 4.

```
> data(stockdata) #Load the stock data
> Y = log(stockdata$data[2:1258,]/stockdata$data[1:1257,]) #Preprocessing
> out = fastclime(Y,0.06) #Estimate the graph
> plot(out)
> out1 = fastclime.lambda(out$lambdaTx, out$icovlist, 0.08)
> fastclime.plot(out1$path)
```

## Sparsity vs. Regularization

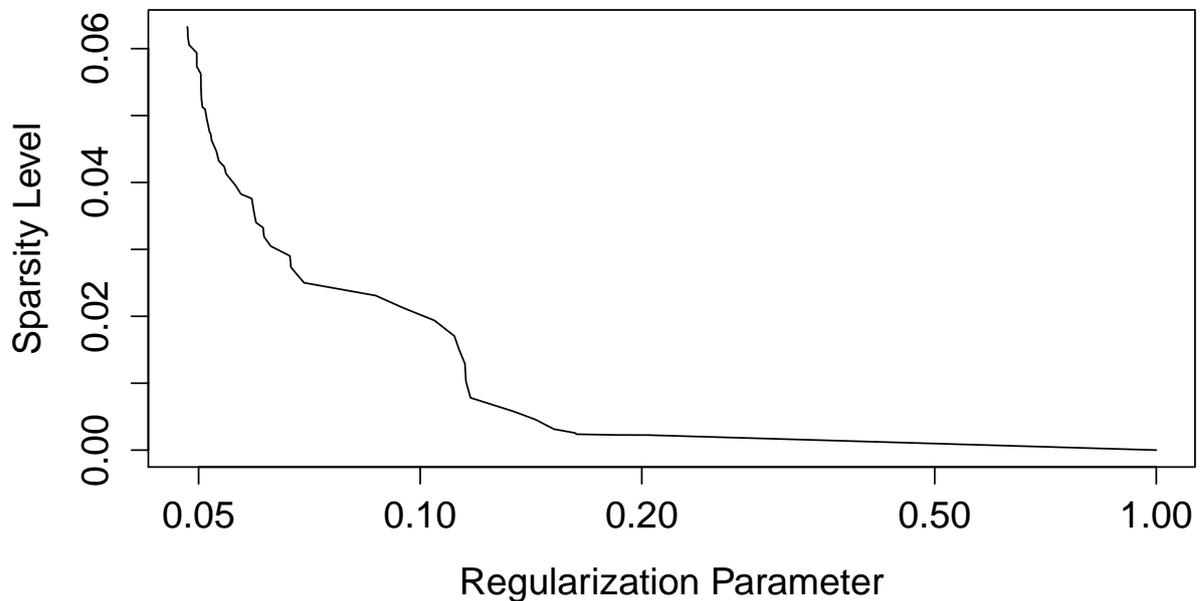


Figure 3: The solution path of the stock data

The linear programming solver function `fastlp()` is used to solve a general linear programming problem in standard inequality form (1). If the original problem is not in this form, the user has to convert it into this form. For example, if the original problem has an equation, then this equation can be separated into two inequalities. If the original variables are not nonnegative, then the trick mentioned in last section to derive (19) can be used to make sure the variables be nonnegative.

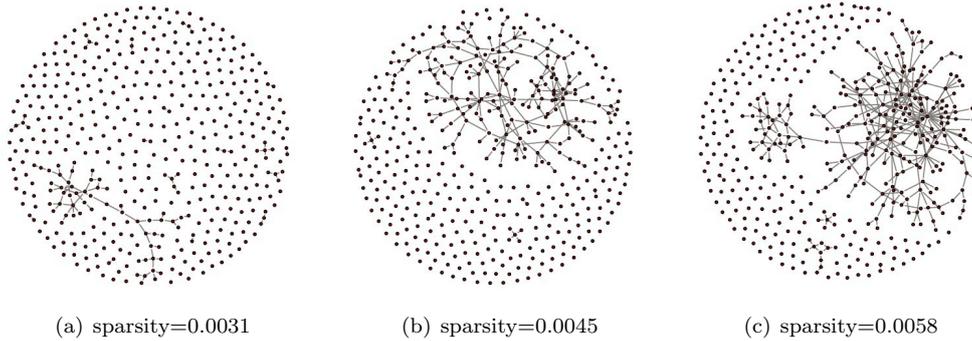


Figure 4: Graphs estimated by fastclime based on the stock data

`fastlp()` has four parameters. The first three are the objective vector (length  $n$ ), the constraint matrix (dimension  $m \times n$ ) and the right hand side vector (length  $m$ ). The last parameter `lambda` is used to specify the stopping criteria of the linear programming solver. Whenever the calculated parameter in the solver is smaller than this value, the program will stop at that point. The default value of the `lambda` is zero, which corresponds to the optimal value of the LP problem. The function will also indicate if the input problem is unbounded or infeasible. Another linear programming solver function `paralp()` is used to solve a parameterized LP problem in form (9). Notice the perturbation vectors must be nonnegative. Here is some examples: we generate a random feasible LP problem and solves it, then we give a simple LP examples in form (9) and solves it.

```
> n=100
> A=matrix(mvnrnorm(n^2,0, 1),nrow=n)
> x=runif(n, min = 0, max = 5)
> y=runif(n, min = 0, max = 5)
> w=runif(n, min = 0, max = 5)
> z=runif(n, min = 0, max = 5)
> b=A%*%x+w
> c=t(A)%*%y-z
> out1<-fastlp(c,A,b)
> c<-c(1,2,1,1)
> b<-c(8,12,18)
> A<-matrix(c(2,2,3,1,2,1,5,0,2,1,4,0),nrow=4)
> b_bar<-c(1,1,1)
> c_bar<-c(1,1)
> opt2<-paralp(c,A,b,c_bar,b_bar)
```

## 4 Performance Benchmark

Since CLIME has many advantages over maximum likelihood estimator (MLE), we focus on the comparison between packages based on CLIME method. We evaluate the timing performance of our package with comparison to the packages `flare` and `clime` while estimating the precision matrix. `flare` [Li et al., 2012] uses Alternating Direction Method of Multiplier (ADMM) as the method to evaluate CLIME and `clime` solves a series of LP problems for different values of  $\lambda$ . We simulate the data from several multivariate normal distributions. We fix the sample size  $n$  to be 200 and vary the data dimension  $d$  from 50 to 800. We generate our data by `fastclime.generator`, without any particular data structure.

The error tolerance (primal and dual gap) used for `clime` and `fastclime` is  $1 \times 10^{-5}$  and the error tolerance (differences between iterations) used for `flare` is also  $1 \times 10^{-5}$ . Our package calculates its own  $\lambda$  sequences while solving the LP only once, the  $\lambda$  sequence is stored in the output. These  $\lambda$  sequences used for the other two methods are all the default sequences from the packages. The  $\lambda$  sequence used by `clime` is of length 100 from 0.8 to 100 and the sequence

used by `flare` is of length 10 from  $0.5\lambda_{\max}$  to  $\lambda_{\max}$ . The default value of  $\lambda_{\max}$  is the minimum regularization parameter, which yields an all-zero off-diagonal estimates.

As can be seen from Table 1, `fastclime` performance significantly faster than `clime` when  $d$  is 50 or 100. When  $d$  is large, we are not able to obtain results directly from `clime` in one hour. We also notice that, in most cases, `fastclime` performances better than `flare`, and it has a smaller deviation compared with `flare`. The units are in seconds and all experiments are carried on a PC with Intel Core i5-3320 2.6GHz processor and the R used is 2.15.0.

Table 1: Average Timing Performance of Three Solvers

| solve     | d=50          | d=100         | d=200         | d=400          | p=800            |
|-----------|---------------|---------------|---------------|----------------|------------------|
| clime     | 103.52(9.11)  | 937.37(6.77)  | N/A           | N/A            | N/A              |
| flare     | 0.632(0.335)  | 1.886(0.755)  | 10.770(0.184) | 74.106(33.940) | 763.632(135.724) |
| fastclime | 0.248(0.0148) | 0.928(0.0268) | 9.928(3.702)  | 53.038(1.488)  | 386.880(58.210)  |

## 5 Conclusions

We developed a new package named `fastclime` for generic and parameterized linear programming problems as well as conditional independence graph estimation and precision matrix recovery. The package is based on CLIME and the parametric simplex method. Compared with the existing package `clime`, it has many additional features: it is much faster and it can recover the full piece-wise linear regularization solution path. We hope the CRAN community could benefit from our contributions. We plan to maintain and support this package in the future.

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