# Some details on the Delta mehod to obtain standard errors and covariances of item parameters

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April 7th, 2013

This note provides a complementary information to the help files of the functions `itemPar2PL()`, `itemPar3PL()` and `itemPar3PLconst()` from the R package `difR`.

## 1 IRT models

The basic IRT model under consideration is the three-parameter logistic(3PL) model:

$$Pr(X_i = 1|\theta) = c_i + \frac{1 - c_i}{1 + \exp\left\{-[a_i\,(\theta - b_i)]\right\}}, \tag{1}$$

where $X_i$ is the binary response to item $i$ (coded as zero for an incorrect response and one as a correct response), $\theta$ is the ability level, and $a_i$, $b_i$ and $c_i$ are respectively the discrimination, difficulty and pseudo-guessing parameters of the item $i$.

The function `itemPar3PL()` aims at calibrating the 3PL model (1) and providing estimates of the item parameters, related standard errors, and covariances between item parameters. The function `tpm()` of the R package `ltm` is used.

The function `itemPar3PLconst()` fits the 3PL model (1) but by constraining the pseudo-guessing parameters $c_i$ to pre-specified values. In other words, in the constrained 3PL model only the $a_i$ and $b_i$ parameters are estimated.

Finally, the `itemPar2PL()` function fits the two-parameter logistic model, derived from (1) by fixing all $c_i$ parameters to zero:

$$Pr(X_i = 1|\theta) = \frac{1}{1 + \exp\left\{-[a_i\,(\theta - b_i)]\right\}} = \frac{\exp\left\{a_i\,(\theta - b_i)\right\}}{1 + \exp\left\{a_i\,(\theta - b_i)\right\}}. \tag{2}$$

The calibration is performed through the `ltm()` function of the eponym package.

## 2 Linear parametrization

Although models (1) and (2) are written in their IRT form, the R package `ltm` fits actually the linear parametrized versions of those models, that is:

$$Pr(X_i = 1|\theta) = \gamma_i + \frac{1 - \gamma_i}{1 + \exp\{-[\beta_i + \alpha_i\,\theta]\}} \tag{3}$$

and

$$Pr(X_i = 1|\theta) = \frac{\exp\{\beta_i + \alpha_i\,\theta\}}{1 + \exp\{\beta_i + \alpha_i\,\theta\}}. \tag{4}$$

Consequently, `ltm` returns estimate of model parameters $\alpha_i$, $\beta_i$ and $\gamma_i$, as well as related covariance matrix from which the standard errors can be extracted.

Of interest is to obtain similar values for the IRT parameters $a_i$, $b_i$ and $c_i$. It is straightforward to notice the relationships between the model parameters in (3) and (4) and their IRT counterparts in (1) and (2):

$$\left\{\begin{array}{lcl} a_i & = & \alpha_i \\ b_i & = & -\beta_i/\alpha_i \\ c_i & = & \gamma_i. \end{array}\right. \tag{5}$$

## 3 Delta method

The derivation of standard errors and covariances between item parameters $a_i$, $b_i$ and $c_i$ is performed through the Delta method. The general principle is described hereafter. Specific formulas for the present framework are derived in the next section.

Set $p$ as the number of parameters per item and let $(B_{i1}, ..., B_{ip})$ be these parameters for item $i$. For ease of notation, the subscript $i$ will be removed from this section. From the estimation routine, both item parameter estimates and the related covariance matrix can be computed. For any set of parameters $B_j$ and $B_k$ $(j, k = 1, ..., p)$, set $\Sigma_j = Var(B_j)$ and $\Sigma_{jk} = Cov(B_j, B_k)$.

Now, assume that the item parameters $(B_1, ..., B_p)$ are transformed onto a new set of parameters $(C_1, ..., C_p)$ by means of some well-defined functions $h_j$:

$$C_j = h_j(B_1, ..., B_p), \qquad j = 1, ..., p. \tag{6}$$

Getting parameter estimates for $(C_1, ..., C_p)$ are directly obtained by applying the corresponding transformations $(h_1, ..., h_p)$ to the "original" parameters $(B_1, ..., B_p)$, while the elements of the covariance matrix can be obtained by the Delta method. The full calculations are not reported here, but below are the operating functions to obtain the variances of the parameters $C_j$ and the covariances between $C_j$ and $C_k$:

$$Var(C_j) = \sum_{t=1}^{p} \left(\frac{\partial\, h_j}{\partial\, B_t}\right)^2 \Sigma_t + \sum_{t=1}^{p} \sum_{s \neq t} \left(\frac{\partial\, h_j}{\partial\, B_t}\right) \left(\frac{\partial\, h_j}{\partial\, B_s}\right) \Sigma_{ts} \tag{7}$$

and

$$Cov(C_j, C_k) = \sum_{t=1}^{p} \left(\frac{\partial\, h_j}{\partial\, B_t}\right) \left(\frac{\partial\, h_k}{\partial\, B_t}\right) \Sigma_t + \sum_{t=1}^{p} \sum_{s \neq t} \left(\frac{\partial\, h_j}{\partial\, B_t}\right) \left(\frac{\partial\, h_k}{\partial\, B_s}\right) \Sigma_{ts}, \tag{8}$$

where $h_j = h_j(B_1, ..., B_p)$.

# 4 Application to IRT models

As explained earlier, the item parameters $\alpha_i$, $\beta_i$ and $\gamma_i$ of the linear parametrization, and the related covariance matrix are obtained by the estimation routines of the `ltm` package. The IRT parameters $a_i$, $b_i$ and $c_i$ are defined as follows:

$$\left\{ \begin{array}{lcl} b_i & = & h_1(\beta_i, \alpha_i, \gamma_i) = -\beta_i / \alpha_i \\ a_i & = & h_2(\beta_i, \alpha_i, \gamma_i) = \alpha_i \\ c_i & = & h_3(\beta_i, \alpha_i, \gamma_i) = \gamma_i. \end{array} \right. \tag{9}$$

It comes then

$$\begin{array}{ccc} \dfrac{\partial\, h_1}{\partial\, \alpha_i} = \dfrac{\beta_i}{{\alpha_i}^2}, & \dfrac{\partial\, h_2}{\partial\, \alpha_i} = 1, & \dfrac{\partial\, h_3}{\partial\, \alpha_i} = 0, \\ \dfrac{\partial\, h_1}{\partial\, \beta_i} = \dfrac{-1}{\alpha_i}, & \dfrac{\partial\, h_2}{\partial\, \beta_i} = 0, & \dfrac{\partial\, h_3}{\partial\, \beta_i} = 0, \\ \dfrac{\partial\, h_1}{\partial\, \gamma_i} = 0, & \dfrac{\partial\, h_2}{\partial\, \gamma_i} = 0, & \dfrac{\partial\, h_3}{\partial\, \gamma_i} = 1. \end{array} \tag{10}$$

Defining eventually

$$\Sigma_{i,1} = Var(\beta_i), \quad \Sigma_{i,2} = Var(\alpha_i), \quad \Sigma_{i,3} = Var(\gamma_i), \tag{11}$$

and

$$\Sigma_{i,12} = Cov(\alpha_i, \beta_i), \quad \Sigma_{i,13} = Cov(\alpha_i, \gamma_i), \quad \Sigma_{i,23} = Cov(\beta_i, \gamma_i), \tag{12}$$

it comes that the formulas (7) and (8) can be written in this context as follows:

$$\left\{ \begin{array}{l} Var(a_i) = Var[h_2(\beta_i, \alpha_i, \gamma_i)] = \Sigma_{i,2} \\ Var(b_i) = Var[h_1(\beta_i, \alpha_i, \gamma_i)] = \frac{\Sigma_{i,1}}{{\alpha_i}^2} + \frac{{\beta_i}^2\, \Sigma_{i,2}}{{\alpha_i}^4} - \frac{2\, \beta_i\, \Sigma_{i,12}}{{\alpha_i}^3} \\ Var(c_i) = Var[h_3(\beta_i, \alpha_i, \gamma_i)] = \Sigma_{i,3} \end{array} \right. \tag{13}$$

and

$$\begin{cases} Cov(a_i, b_i) = \frac{\beta_i \, \Sigma_{i,2}}{{\alpha_i}^2} - \frac{\Sigma_{i,12}}{\alpha_i} \\ Cov(a_i, c_i) = \Sigma_{i,23} \\ Cov(b_i, c_i) = \frac{\beta_i \, \Sigma_{i,13}}{{\alpha_i}^2} - \frac{\Sigma_{i,13}}{\alpha_i}. \end{cases} \tag{14}$$

Consequently, the standard errors of the item parameters $a_i$, $b_i$ and $c_i$ are computed as the square roots of the formulas (13), and the covariances are computed directly from (14). Incase of the constrained 3PL model or the 2PL model, all output information regarding the $c_i$ parameters are discarded but the formulas for parameters $a_i$ and $b_i$ are left unchanged.