Log-likelihood equations for LEXPIT model

Under the LEXPIT, the model for the binomial probability is

$$\pi = x'\beta + \text{expit}(z'\gamma)$$

for x and z covariates, with z including an intercept term and 'expit' denotes the sigmoid (inverse-logit) function $\text{expit}(x) = exp(x)/(1 + exp(x))$.

For n observations and the ith event outcome as y_i and case weight w_i, the score for an arbitrary coefficient θ is

$$S(\theta) = \sum_i \dot{\pi}(\theta; x_i, z_i) w_i \nu(\pi(\theta; x_i, z_i))^{-1}(y_i - \pi(\theta; x_i, z_i)),$$

where $\nu(x) = x(1-x)$ the first derivative of the expit function. Let $A_i(\theta) = \nu(\pi(\theta; x_i, z_i))^{-1}(y_i - \pi(\theta; x_i, z_i))$. The Hessian is

$$\dot{S}(\theta) = \sum_i w_i \left[\ddot{\pi}(\theta; x_i, z_i) A_i(\theta) - \dot{\pi}(\theta; x_i, z_i)\dot{\pi}(\theta; x_i, z_i)' \nu(\pi(\theta; x_i, z_i))^{-1}(A_i(\theta) + 1)\right].$$

For $\pi(\beta; x_i, z_i)$,

$$\dot{\pi}(\beta; x_i, z_i) = x_i \text{ and } \ddot{\pi}(\beta; x_i, z_i) = 0.$$

For $\pi(\gamma; x_i, z_i)$,

$$\dot{\pi}(\gamma; x_i, z_i) = z_i \text{dexpit}(z_i'\gamma)$$

and

$$\ddot{\pi}(\gamma; x_i, z_i) = z_i z_i' \text{ddexpit}(z_i'\gamma),$$

where $\text{dexpit}(x) = \text{expit}(x)(1 - \text{expit}(x))$ and $\text{ddexpit}(x) = \text{dexpit}(x)(1 - 2\text{expit}(x))$. The Hessian for β is

$$\dot{S}(\beta) = -\sum_i w_i x_i x_i' \nu(\pi(\theta; x_i, z_i))^{-1}(A_i(\theta) + 1).$$

and for γ is

$$\dot{S}(\gamma) = \sum_i w_i z_i z_i' \left[\text{ddexpit}(z_i'\gamma) A_i(\theta) - \text{dexpit}(z_i'\gamma)^2 \nu(\pi(\theta; x_i, z_i))^{-1}(A_i(\theta) + 1)\right],$$

and the partial derivative of $S(\beta)$ with respect to γ is

$$\frac{\partial S(\beta)}{\partial \gamma} = -\sum_i w_i x_i z_i' \left[\text{dexpit}(z_i'\gamma)\nu(\pi(\theta; x_i, z_i))^{-1}(A_i(\theta) + 1)\right].$$

In matrix form,

$$\mathcal{H}(\beta) = -X'W(\beta)X$$

with $W(\beta) = \mathrm{Diag}\{w_1 \nu(\pi(\theta; x_1, z_1))^{-1}(A_1(\theta) + 1), \ldots\}$.

$$\mathcal{H}(\gamma) = Z'W(\gamma)Z$$

with $W(\gamma) = \mathrm{Diag}\{w_1 \mathrm{ddexpit}(z_1'\gamma)A_1(\theta) - \mathrm{dexpit}(z_1'\gamma)^2 \nu(\pi(\theta; x_1, z_1))^{-1}(A_1(\theta) + 1), \ldots\}$.

$$\mathcal{H}(\beta, \gamma) = -X'W(\beta, \gamma)Z$$

with $W(\beta, \gamma) = \mathrm{Diag}\{w_1 \mathrm{dexpit}(z_1'\gamma)\nu(\pi(\theta; x_1, z_1))^{-1}(A_1(\theta) + 1), \ldots\}$.