Contents

Adjusting Likelihood Ratio Confidence Intervals for Parameters Near Boundaries Applied to the Binomial

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Problem Statement

- **Interval estimation on p is not simple and there seems to be no agreement on which is best**
- **Intervals when there are 0% or 100% passes tend to be too short**
  - A standard adjustment for “parameter at a boundary” assumes 2 * log(likelihood ratio) is a mixture of chi-squares
  - For binomial confidence intervals, this is equivalent to using $\chi^2_{1-\alpha/2}$ in place of $\chi^2_{1-\alpha}$
  - How well does this work?
  - Can we find something better that is almost as simple?

Binomial Log-Likelihood

- **The binomial log-likelihood is given by**

$$\ell(p, x, n) = \log \binom{n}{x} + x \log(p) + (n - x) \log(1 - p)$$

where n is the number of independent Bernoulli trials, x is the number of successes out of n, and p is the probability of success

- **The Maximum Likelihood Estimate (MLE) of p is given by**

$$\widehat{p} = \frac{x}{n}$$

Confidence Intervals

- **Interval estimates of p are difficult to achieve due to the discreteness and skewness (for $p \neq 0.5$) of the binomial distribution**
- **Many methods have been devised to estimate confidence intervals on p**
  - Likelihood methods: generalized linear models, likelihood ratio, asymptotic
  - Bayesian
  - Inversion methods: Wilson, Agresti-Coulls, Fleiss, Clopper-Pearson
- **The asymptotic method is woefully poor but still part of most standard statistics curricula**

Confidence Intervals Based On Likelihood Ratio

- **The likelihood ratio test statistic is define by**

$$\Lambda(p_0, \widehat{p}, x, n) = \ell(\widehat{p}, x, n) - \ell(p_0, x, n) \sim \chi_1^2,$$

where

$$\widehat{p} = \frac{x}{n}$$

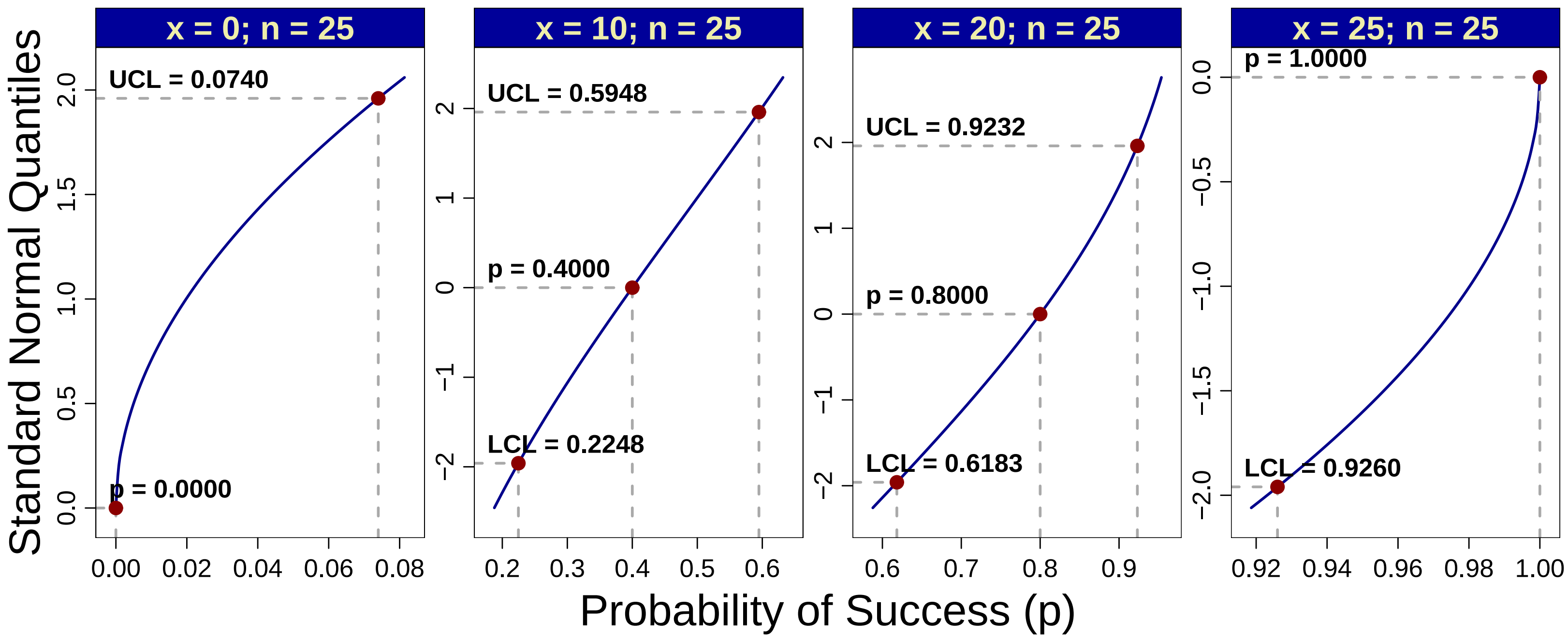
is the MLE and p_0 is the probability of success under the null hypothesis

- **Inverting L, we obtain a confidence interval on p:**

$$\begin{aligned} LCL &= \arg\max_{0<p<1} \left\{ \Lambda(p, \widehat{p}, x, n) - 0.5\chi_1^2 \right\} \\ UCL &= \arg\min_{0<p<1} \left\{ \Lambda(p, \widehat{p}, x, n) - 0.5\chi_1^2 \right\} \end{aligned}$$

Estimating The Likelihood Ratio Confidence Interval

- **The method requires an iterative root-finding algorithm to find the lower and upper confidence bound**



- **We will refer to this interval estimate as “LRT”**

Coverage Probability

- **Coverage probability determines the expected value of any interval estimate over the binomial density function**

$$C(p, x, n) = \sum_{x=0}^{n} I(LCL < p < UCL) \binom{n}{x} p^x (1-p)^{n-x}$$

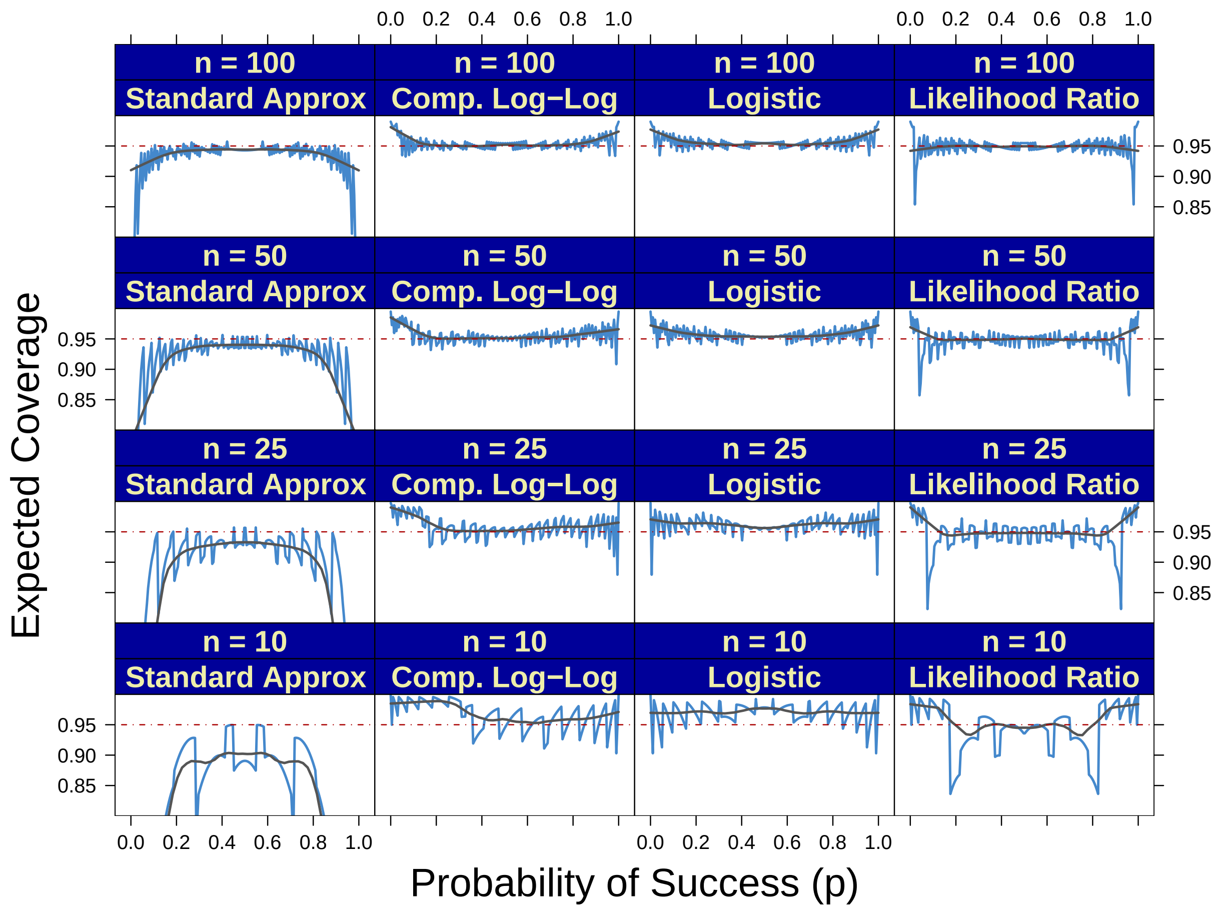
where p is the true probability of success, and (LCL, UCL) is an interval estimate of p

Properties of $C(p, x, n)$

- **Should be close to the level of confidence** $(1 - \alpha)$
- **Oscillates due to the discreteness and skewness of the binomial distribution**
- **There are** $2 \cdot n$ **discontinuities (jumps) which exist at the each confidence interval endpoint**

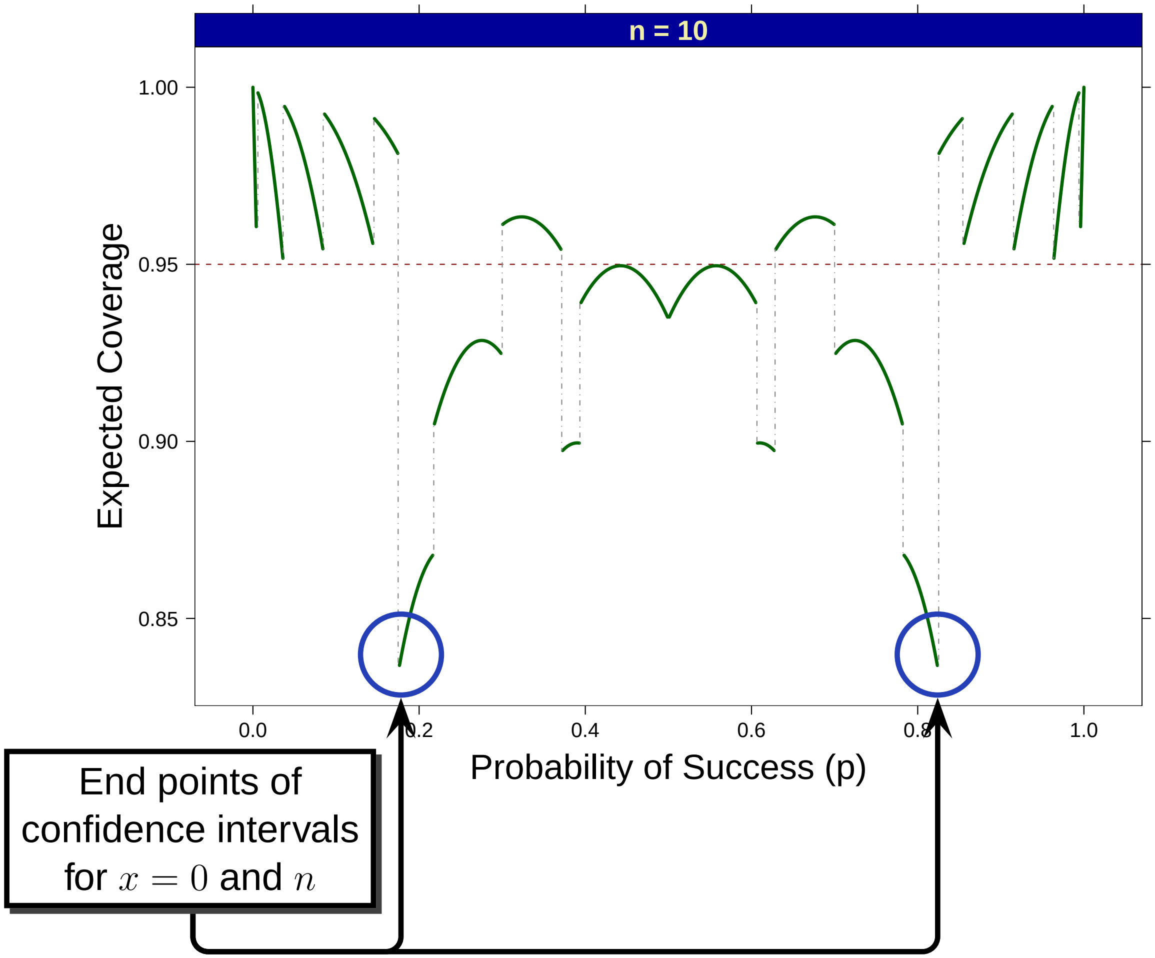
Coverage Probability For Several Methods

- **The LRT method has the best coverage probability**
  - The standard (asymptotic) method is absolutely the worst
  - The compli-mentary log-log is not symmetrical



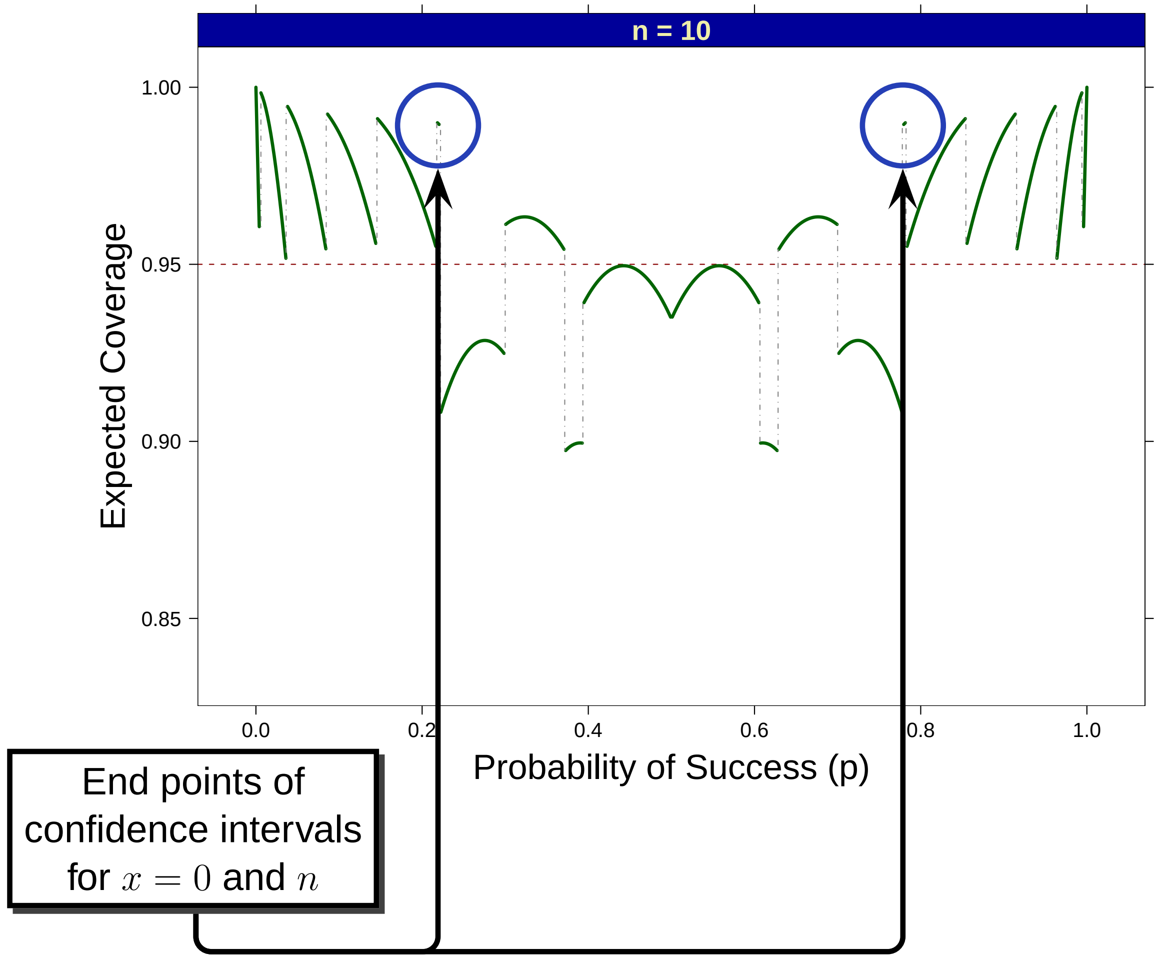
Confidence Intervals At The Boundaries

- **Coverage of intervals when $x = 0$ or n is not optimal**
  - Expected coverage is much less than $1 - \alpha$ for p close to the interval end
  - Adjusting the α downward improves coverage by increasing the interval length



Adjusting Confidence Intervals At The Boundaries

- **Changing the signficance probability from 0.05 to 0.025 improves the coverage**
  - Still not optimal as the discontinuity is too large
  - Coverage is too high because length of intervals when $x \neq 0$ or n seem to be too long



Optimal Coverage

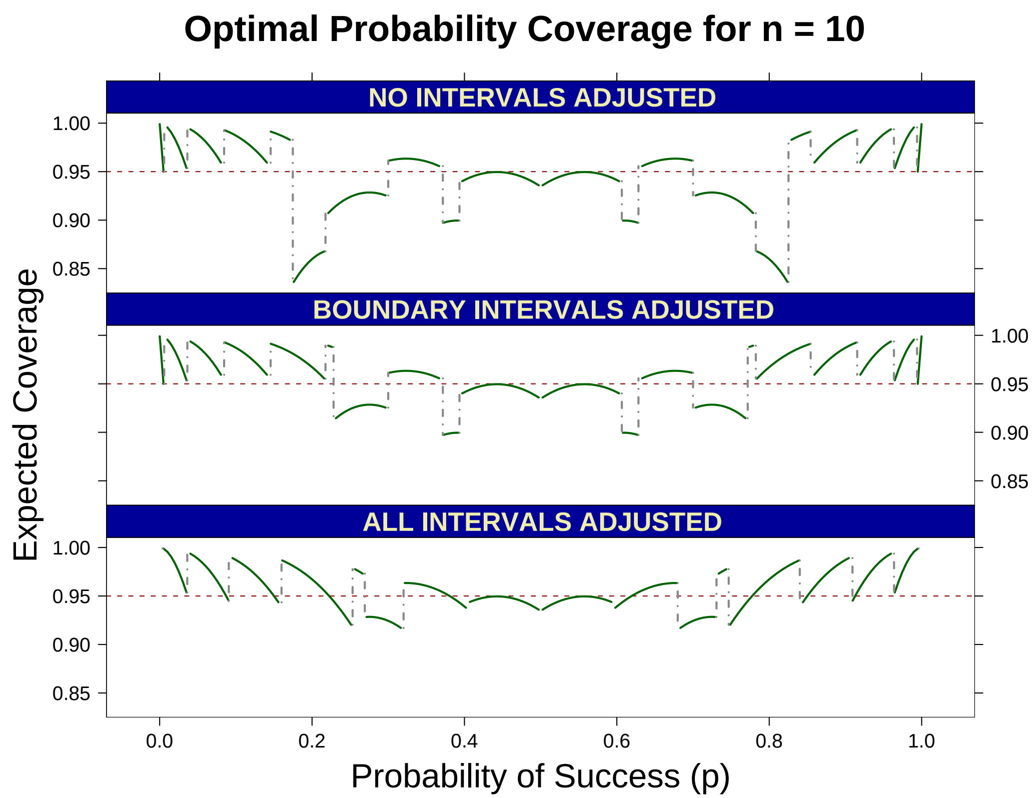
- **Minimize the squared area between the expected coverage and the desired level of confidence**

$$\alpha_0 = \underset{0<\alpha<1}{\arg\min} \int_0^1 \left[C(p,x,n) - (1-\alpha)\right]^2 dp$$

- **Minimizing latter objective function can be achieved by adjusting α for all x or simply for $x = 0$ and n**
  - Adjusting only the boundary intervals is computationally fairly fast for relatively small n
  - Adjusting all the intervals can be slow even for small n

Coverage Using Optimal "α_0"

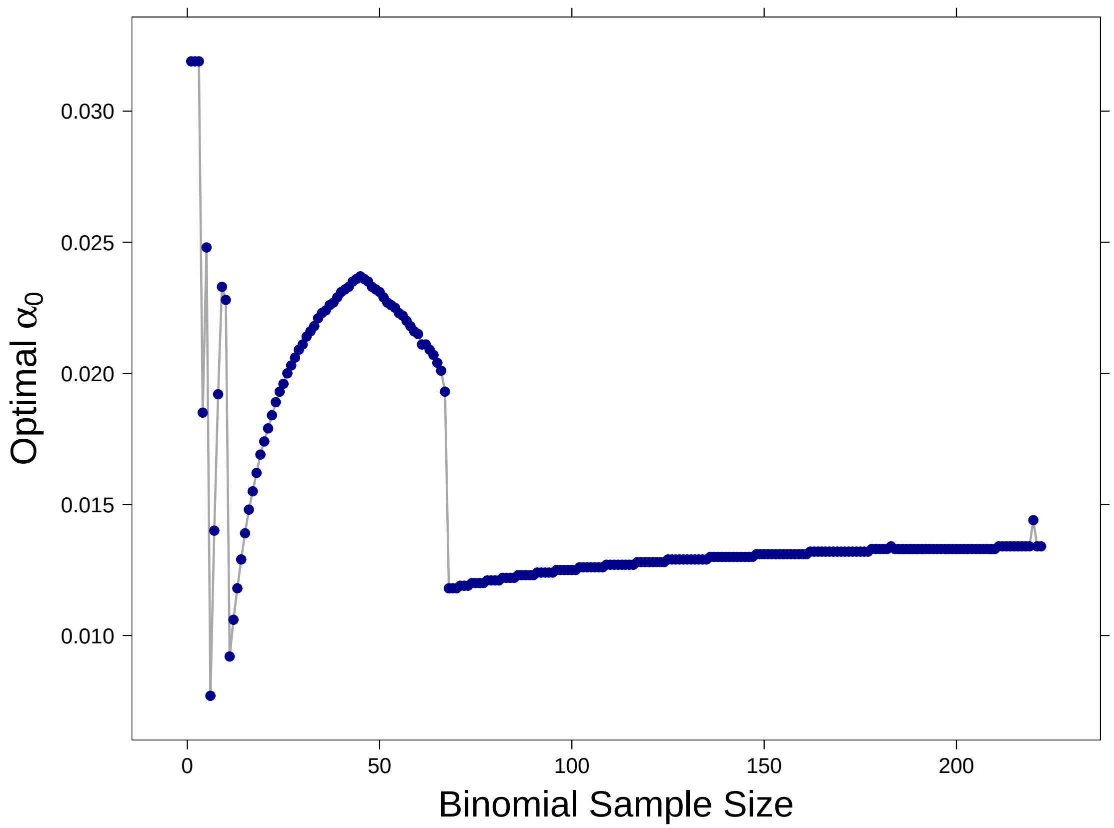
- **Using an optimal "α_0" improves coverage**
- **Example with $n = 10$**
  - Adjusting boundary intervals only, α_0 is 0.023 when $x = 0$ or 10
  - Adjusting all intervals, α_0 is 0.012 for the boundary intervals but monotonically increasing to 0.095 when $x = 5$



x	0	1	2	3	4	5
α_0	0.012	0.030	0.050	0.061	0.070	0.095

Adjustments To Significance Probabilities

- **Optimal confidence level asymptotes around 0.14 for $x = 0$ or n**



Summary

- **Using the LRT confidence interval produces the best coverage but are not computable by hand**
- **Confidence intervals for p when the observed number of successes are close to 0 or n are too short using a constant level of confidence**
- **There is no solution independent of n for adjusting a confidence intervals at the boundaries**
- **Final recommendation:**
  - Use the LRT confidence interval (see next slide for software)
  - For $x = 0$ or n set α between 0.015 and 0.025
  - For obtaining all confidence interval adjustments use the `binom` package in R

The `binom` package

- **An R package for constructing confidence intervals on the probability of success in a binomial experiment via several parameterizations**
  - Bayes, LRT, probit, logit, cloglog
  - Coverage plotting
  - Optimal coverage
  - Sample size calculation and Power curves
  - Tcl/Tk interface for Power curves

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