Clifford algebra in R

Robin K. S. Hankin
Auckland University of Technology

Abstract

Here I present the **clifford** package for working with Clifford algebras. The algebra is described and package idiom is given.

Keywords: Clifford algebra.

1. Introduction

Clifford algebras are an interesting and instructive mathematical object with a rich structure that has varied applications to physics.

1.1. Existing work

Computational support for working with the Clifford algebras is part of a number of algebra systems including Sage (The Sage Developers 2019) and `sympy` (Meurer *et al.* 2017). Here I introduce the **clifford** package, which provides R-centric functionality for Clifford algebras. Notation follows Snygg (2010).

Considering a vector space of dimension 3, and given a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, we can consider linear combinations such as

$$\begin{aligned}\mathbf{x} &= x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 \\ \mathbf{y} &= y^1\mathbf{e}_1 + y^2\mathbf{e}_2 + y^3\mathbf{e}_3.\end{aligned} \tag{1}$$

A Clifford algebra includes a formal product on such sums, defined using the relations

$$(\mathbf{e}_1)^2 = (\mathbf{e}_2)^2 = (\mathbf{e}_3)^2 = 1 \tag{2}$$

$$\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_2 = 0 \tag{3}$$

This gives:

$$\begin{aligned}\mathbf{xy} &= \left(x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3\right)\left(y^1\mathbf{e}_1 + y^2\mathbf{e}_2 + y^3\mathbf{e}_3\right) \\ &= \left(x^1y^1 + x^2y^2 + x^3y^3\right) + \\ &\quad \left(x^2y^3 - x^3y^2\right)\mathbf{e}_2\mathbf{e}_3 + \left(x^3y^1 - x^1y^3\right)\mathbf{e}_1\mathbf{e}_3 + \left(x^1y^2 - x^2y^1\right)\mathbf{e}_1\mathbf{e}_2\end{aligned} \tag{4}$$

Multiplication is associative by design. Snygg goes on to consider the algebra spanned by products of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and shows that this is an eight dimensional space spanned by

$$\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{31}, \mathbf{e}_{12}, \mathbf{e}_{123}\} \tag{5}$$

where $\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2$ and so on. Thus a general element of this space would be

$$a^0 + a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3 + a^{12}\mathbf{e}_{12} + a^{31}\mathbf{e}_{31} + a^{23}\mathbf{e}_{23} + a^{123}\mathbf{e}_{123} \tag{6}$$

(here the a's are real). That the space is closed under multiplication follows from equations 2 and 3; thus, for example,

$$\mathbf{e}_1\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_1\mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_3\mathbf{e}_2 = -\mathbf{e}_3\mathbf{e}_2 = -\mathbf{e}_{23}. \tag{7}$$

(observe how associativity is assumed).

1.2. Generalization to arbitrary dimensions

Generalization to higher dimensional vector spaces is easy. Suppose we consider a nine-dimensional vector space spanned by $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Then an arbitrary vector in this space will be $a^1\mathbf{e}_1 + \cdots + a^n\mathbf{e}_n$. The associated Clifford algebra will be of dimension 2^n, spanned by elements like $\mathbf{e}_1\mathbf{e}_3\mathbf{e}_5 = \mathbf{e}_{135}$ and $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_5 = \mathbf{e}_{1235}$. The defining relations would be

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 2n_{ij} \tag{8}$$

where

$$n_{ij} = \begin{cases} 1, & i = j \\ 0 & i \neq j \end{cases} \tag{9}$$

1.3. Clifford algebra in a pseudo-Euclidean space

Equations 8 and 9 defined a positive-definite inner product on the vector space spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. This is readily generalized to allow a more general inner product. Conventionally we define

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 2n_{ij} \tag{10}$$

where

$$n_{ij} = \begin{cases} 1, & i = j = 1, \ldots, p \\ -1, & i = j = p+1, \ldots, n \\ 0, & i \neq j \end{cases} \tag{11}$$

for $1 \leqslant p \leqslant n$; usually we also specify $p + q = n$ and write $\mathbb{R}_{p,q}$ (sometimes $\mathcal{C}_{p,q}$) for the Clifford algebra specified by equations 10 and 11. Annoyingly enough, the metric tensor η as used in relativity (usually) has the negative sign first:

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

1.4. Wedge product of the exterior algebra is a special case of the geometric product

If we specify that the quadratic form is identically zero then equation 10 becomes

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 0, \qquad 1 \leqslant i, j \leqslant p \tag{13}$$

which implies that $\mathbf{e}_i\mathbf{e}_i = 0$. Geometric products become wedge products (although linearity means that we may add terms of different grades, unlike conventional Grassman algebra).

2. The package in use

Suppose we want to work with arbitrary Clifford object $1 + 2\mathbf{e}_1 + 3\mathbf{e}_2 + 4\mathbf{e}_2\mathbf{e}_3$. In R idiom this would be

```
> (x <- clifford(list(numeric(0),1,2,2:3),1:4))

Element of a Clifford algebra, equal to
+ 1 + 2e_1 + 3e_2 + 4e_23
```

Function `clifford()` takes a list of terms and a vector of coefficients. Addition and subtraction work as expected:

```
> y <- clifford(list(1),2)
> x-y

Element of a Clifford algebra, equal to
+ 1 + 3e_2 + 4e_23
```

In the above, see how the $\mathbf{e}_1$ term has vanished. We can multiply Clifford elements using natural R idiom:

```
> x*x

Element of a Clifford algebra, equal to
- 2 + 4e_1 + 6e_2 + 8e_23 + 16e_123
```

(Multiplication that Snygg denotes by juxtaposition is here indicated with a `*`). We can consider arbitrarily high dimensional data:

```
> (z <- as.1vector(1:7))

Element of a Clifford algebra, equal to
+ 1e_1 + 2e_2 + 3e_3 + 4e_4 + 5e_5 + 6e_6 + 7e_7

> z*x

Element of a Clifford algebra, equal to
+ 8 + 1e_1 - 10e_2 - 1e_12 + 11e_3 - 6e_13 - 9e_23 + 4e_123 + 4e_4 - 8e_14 -
12e_24 + 16e_234 + 5e_5 - 10e_15 - 15e_25 + 20e_235 + 6e_6 - 12e_16 - 18e_26 +
24e_236 + 7e_7 - 14e_17 - 21e_27 + 28e_237
```

In the above, we coerce a vector to a Clifford 1-vector. The package includes many functions to generate Clifford objects:

```
> rcliff()

Element of a Clifford algebra, equal to
+ 9e_3 + 8e_124 + 7e_134 + 3e_5 + 6e_25 + 5e_1235 + 1e_6 + 2e_36 + 4e_1246
```

The defaults for `rcliff()` specify that the object is a sum of grade-4 terms but this can be altered:

```
> (x <- rcliff(d=7,g=5,include.fewer=TRUE))

Element of a Clifford algebra, equal to
+ 7e_5 + 8e_6 + 3e_1246 + 4e_2356 + 9e_12456 + 5e_13456 + 1e_27 + 2e_2357 +
6e_457

> grades(x)

[1] 1 1 4 4 5 5 2 4 3
```

2.1. Pseudo-Euclidean spaces

The signature of the metric may be altered. Starting with the Euclidean case we have:

```
> e1 <- basis(1)
> e2 <- basis(2)
> e1*e1

Element of a Clifford algebra, equal to
scalar ( 1 )

> e2*e2
```

```
Element of a Clifford algebra, equal to
scalar ( 1 )
```

(package idiom `basis(i)` returns $\mathbf{e}_i$). However, if we wish to consider $n = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, the package idiom is to use the `signature()` function:

```
> signature(1)  # signature +-

[1] 1

> e1*e1 # as before

Element of a Clifford algebra, equal to
scalar ( 1 )

> e2*e2 # sign changes

Element of a Clifford algebra, equal to
scalar ( -1 )
```

Suppose we wish to use a signature $+++-$, corresponding to the Minkowski metric in special relativity; this would be indicated in package idiom by `signature(3)`. Note that the clifford objects themselves do not store the signature; it is used only by the product operation `*`.

```
> x <- rcliff(d=4,g=3,include.fewer=TRUE)
> y <- rcliff(d=4,g=3,include.fewer=TRUE)
```

Then we may multiply these two clifford objects using either the default positive-definite inner product, or the Minkowski metric:

```
> x*y

Element of a Clifford algebra, equal to
+ 66 - 9e_1 + 54e_2 - 132e_12 + 80e_3 - 106e_13 + 75e_23 + 120e_123 + 6e_4 -
36e_14 + 189e_24 + 54e_124 - 26e_34 - 73e_134 + 43e_234 + 28e_1234

> signature(3)  # switch to signature +++-

[1] 3

> x*y

Element of a Clifford algebra, equal to
+ 150 + 9e_1 - 72e_2 + 12e_12 - 8e_3 - 176e_13 + 75e_23 + 120e_123 - 6e_4 -
144e_14 + 81e_24 + 54e_124 + 170e_34 + 133e_134 + 43e_234 + 28e_1234
```

In the above, see how the products are different using the two inner products.

2.2. Grassman algebra

A Grassman algebra corresponds to a Clifford algebra with identically zero inner product. Package idiom is to use a negative signature:

```
> signature(-1)  # specify null inner product
```

```
> is.zero(basis(5)^2)     # should be TRUE
```

```
[1] TRUE
```

This is a somewhat clunky way of reproducing the functionality of the **wedge** package. If we have

```
> x <- clifford(list(1:3, c(2,3,7)), 3:4)
> y <- clifford(list(1:3, c(1,4,5), c(4,5,6)), 1:3)
> x %^% y
```

```
Element of a Clifford algebra, equal to
+ 9e_123456 - 8e_123457 - 12e_234567
```

then the **wedge** idiom for this would be:

```
> (x <- as.kform(rbind(1:3,c(2,3,7)),3:4))
```

```
          val
 2 3 7  =   4
 1 2 3  =   3
```

```
> (y <- as.kform(rbind(1:3,c(1,4,5),4:6),1:3))
```

```
          val
 1 2 3  =   1
 1 4 5  =   2
 4 5 6  =   3
```

```
> x %^% y
```

```
                val
 1 2 3 4 5 6  =   9
 2 3 4 5 6 7  = -12
 1 2 3 4 5 7  =  -8
```

2.3. Positive-definite inner product

Function `signature()` takes the special value zero to return to a positive-definite inner product:

```
> signature(0)

[1] 0
```

With this, $\mathbf{e}_i\mathbf{e}_i = 1$ for any i:

```
> basis(53)^2

Element of a Clifford algebra, equal to
scalar ( 1 )
```

2.4. Higher dimensional spaces

Abłamowicz and Fauser (2012) consider high-dimensional Clifford algebras and consider the following two elements of the 1024-dimensional Clifford algebra which we may treat as $\mathcal{C}_{7,3}$ spanned by $\mathbf{e}_1, \ldots, \mathbf{e}_{10}$ and perform a calculation which I reproduce below (although Abłamowicz and Fauser exploited Bott periodicity, a feature not considered here).

Firstly we change the default print method slightly:

```
> options("basissep" = ",")
```

(this separates the subscripts of the basis vectors with a comma, which is useful for clarity if $n > 9$). We then define clifford elements x, y:

```
> (x <- clifford(list(1:3,c(1,5,7,8,10)),c(4,-10)) + 2)

Element of a Clifford algebra, equal to
+ 2 + 4e_1,2,3 - 10e_1,5,7,8,10

> (y <- clifford(list(c(1,2,3,7),c(1,5,6,8),c(1,4,6,7)),c(4,1,-3)) - 1)

Element of a Clifford algebra, equal to
- 1 + 4e_1,2,3,7 - 3e_1,4,6,7 + 1e_1,5,6,8
```

Their geometric product is given in the package as

```
> signature(7)

[1] 7

> x*y
```

```
Element of a Clifford algebra, equal to
- 2 - 4e_1,2,3 - 16e_7 + 8e_1,2,3,7 - 6e_1,4,6,7 - 12e_2,3,4,6,7 + 2e_1,5,6,8 +
4e_2,3,5,6,8 - 10e_6,7,10 - 40e_2,3,5,8,10 - 30e_4,5,6,8,10 + 10e_1,5,7,8,10
```

in agreement with Abłamowicz and Fauser (2012), although the terms appear in a different order.

3. Conclusions and further work

The **clifford** package furnishes a consistent and documented suite of reasonably efficient R-centric functionality. Further work might include closer integration with the `wedge` package (Hankin 2019).

References

Abłamowicz R, Fauser B (2012). "Symbolic Computations in Higher Dimensional Clifford Algebras." 1206.3683.

Hankin RKS (2019). *wedge: The Exterior Calculus*. R package version 1.0-3, URL https://github.com/RobinHankin/wedge.git.

Meurer A, *et al.* (2017). "SymPy: symbolic computing in Python." *PeerJ Computer Science*, **3**, e103. ISSN 2376-5992. doi:10.7717/peerj-cs.103. URL https://doi.org/10.7717/peerj-cs.103.

Snygg J (2010). *A new approach to differential geometry using Clifford's geometric algebra*. Birkhäuser.

The Sage Developers (2019). *SageMath, the Sage Mathematics Software System (Version 8.6)*. URL https://www.sagemath.org.

Affiliation:

Robin K. S. Hankin
Auckland University of Technology
E-mail: hankin.robin@gmail.com